

A stochastic approach to a new type of parabolic variational inequalities

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Abstract

We study the following quasilinear partial differential equation with two subdifferential operators:

$$\begin{cases} \frac{\partial u}{\partial s}(s, x) + (\mathcal{L}u)(s, x, u(s, x), (\nabla u(s, x))^* \sigma(s, x, u(s, x))) \\ \quad + f(s, x, u(s, x), (\nabla u(s, x))^* \sigma(s, x, u(s, x))) \in \partial \varphi(u(s, x)) + \langle \partial \psi(x), \nabla u(s, x) \rangle, \\ \quad (s, x) \in [0, T] \times \text{Dom} \psi, \\ u(T, x) = g(x), \quad x \in \text{Dom} \psi, \end{cases}$$

where for $u \in C^{1,2}([0, T] \times \text{Dom} \psi)$ and $(s, x, y, z) \in [0, T] \times \text{Dom} \psi \times \text{Dom} \varphi \times \mathbb{R}^{1 \times d}$,

$$(\mathcal{L}u)(s, x, y, z) := \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^*)_{i,j}(s, x, y) \frac{\partial^2 u}{\partial x_i \partial x_j}(s, x) + \sum_{i=1}^n b_i(s, x, y, z) \frac{\partial u}{\partial x_i}(s, x).$$

The operator $\partial \psi$ (resp. $\partial \varphi$) is the subdifferential of the convex lower semicontinuous function $\psi : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ (resp. $\varphi : \mathbb{R} \rightarrow (-\infty, +\infty]$).

We define the viscosity solution for such kind of partial differential equations and prove the uniqueness of the viscosity solutions when σ does not depend on y . To prove the existence of a viscosity solution, a stochastic representation formula of Feymann-Kac type will be developed. For this end, we investigate a fully coupled forward-backward stochastic variational inequality.

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1 Introduction

Crandall and Lions introduced the notion of viscosity solution in [8], and in the later work of Crandall, Ishii and Lions [7], they gave a systematically investigation of the viscosity solution for second order partial differential equations (PDEs), which provides a powerful tool to study PDEs and related problems. Pardoux and Peng were the first to give a stochastic interpretation for the viscosity solutions of semilinear PDEs (see [18] and [22]) via their original work on nonlinear backward stochastic differential equations (BSDEs), [17]. This relation between BSDEs and PDEs was investigated by different authors. Let us emphasize that Pardoux and Tang [21] studied the link between the solution of fully coupled forward-backward stochastic differential equations (FBSDEs) and the associated quasilinear parabolic PDEs. El Karoui, Kapoudjian, Pardoux, Peng and Quenez [11] studied reflected BSDEs in one dimensional and by combing it with a forward stochastic differential equation (SDE), they gave a probabilistic interpretation to the viscosity solution of the related obstacle problem for a parabolic PDE. As a generalisation, Cvitanić and Ma [9] studied reflected FBSDEs and used them to give a probabilistic interpretation for the viscosity solution of quasilinear variational inequalities with a Neumann boundary condition.

On the other hand, related with multi-dimensional reflected SDEs and BSDEs, stochastic variational inequalities (SVIs) were considered by Bensoussan and Răşcanu in [5, 6], Asiminoaei and Răşcanu [2] (For more details, the reader is referred to [20]); BSDEs with subdifferential operators (which are called backward stochastic variational inequalities, BSVIs) were studied by Pardoux and Răşcanu [19]. Moreover, the authors of [19] obtained a generalized Feymann-Kac type formula, which gives a probabilistic interpretation for the viscosity solution of parabolic variational inequalities (PVI). Maticiuc, Pardoux, Răşcanu and Zălinescu [13] extended such PVIs to systems of PVIs. In our paper, motivated by [19] and [26], we consider the following type of PVI

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial s}(s, x) + (\mathcal{L}u)(s, x, u(s, x), (\nabla u(s, x))^* \sigma(s, x, u(s, x))) \\ \quad + f(s, x, u(s, x), (\nabla u(s, x))^* \sigma(s, x, u(s, x))) \in \partial \varphi(u(s, x)) + \langle \partial \psi(x), \nabla u(s, x) \rangle, \\ \hspace{25em} (s, x) \in [0, T] \times Dom \psi, \\ u(T, x) = g(x), \quad x \in Dom \psi. \end{array} \right.$$

This type of PVI is new since it is driven by two subdifferential operators, one operating over the state and the other operating in the domain and perturbing the direction of the gradient. We define the viscosity solution of such kind of PVIs and prove the uniqueness of the viscosity solutions when σ does not depend on y . Indeed, by extending and adapting the approaches of Barles, Buckdahn and Pardoux [4] and Cvitanić and Ma [9], we prove the uniqueness of the viscosity solutions in the class of Lipschitz continuous functions. To prove the existence of a viscosity solution, a stochastic representation formula of Feymann-Kac type will be developed. For this end, we investigate the following general fully coupled FBSDEs with subdifferential operators in both the forward and the backward equations,

$$\left\{ \begin{array}{l} dX_t + \partial \psi(X_t) dt \ni b(t, X_t, Y_t, Z_t) dt + \sigma(t, X_t, Y_t, Z_t) dB_t, \\ -dY_t + \partial \varphi(Y_t) dt \ni f(t, X_t, Y_t, Z_t) dt - Z_t dB_t, \quad t \in [0, T], \\ X_0 = x, \quad Y_T = g(X_T). \end{array} \right.$$

We call this kind equations forward-backward stochastic variational inequalities (FBSVIs). Notice that this type of inequalities includes, as a special case, coupled systems composed of a forward and a backward equation, both reflected at the boundary of a closed convex set. Such kind of FBSVIs are worthy of investigation themselves.

As concerns the fully coupled forward-backward stochastic differential equations (FBSDEs), a generalization of non-coupled forward-backward systems studied by Pardoux and Peng in [18] and [22], using the contraction mapping method, Antonelli [1] was the first to prove the existence and the uniqueness for such equations on a small time interval. To show the solvability of FBSDEs on arbitrary time interval, Ma, Protter and Yong [14] introduced the so-called Four-Step-Scheme, which was inspired by the pioneering work of Ma and Yong [16]. In their approach, the study of FBSDE reduces to the problem of a certain parabolic PDE. However, for this approach one needs that the coefficients are deterministic and the diffusion coefficient has to be non-degenerate. Based on this approach, Delarue got more general results in [10]. Without the above conditions, but with a monotonicity assumption, Hu and Peng [12] used the continuation method to prove that the FBSDE has a unique adapted solution. Peng and Wu [24] extended [12] to the multidimensional case, while Yong [25] weakened the monotonicity assumptions. On the other hand, Pardoux and Tang [21] obtained the solvability of the FBSDE under some natural monotonicity conditions different from those in [12] and [25], by using the contraction mapping method. Moreover, they studied the connection between the solution of FBSDEs and associated quasilinear parabolic PDEs. Recently, Zhang [27] introduced a new approach and new general conditions to get the wellposedness of FBSDEs via the induction method and Ma, Wu, Zhang, Zhang [15] found a unified scheme to show the wellposedness of the FBSDEs in a general non-Markovian framework. In the spirit of Pardoux and Tang [21], Cvitanić and Ma [9] studied reflected FBSDEs and used them to give a probabilistic interpretation for the viscosity solution of quasilinear variational inequalities with a Neumann boundary condition.

In our paper, we will prove the existence and the uniqueness for FBSVIs, i.e., for coupled systems composed of a forward SVI and a BSVI. Unlike [9], our FBSVI is more general. Indeed, our FBSVIs cover the case of reflected FBSDEs, where the reflection of the forward as well as the backward equation takes place at the border of closed convex sets. In addition, the backward equation in our case can be multidimensional. Compared with [19], our FBSVI is fully coupled and the forward equation also includes a subdifferential operator, which induce some difficulties. Indeed, we study the penalized FBSDE using Yosida approximation for lower semicontinuous (l.s.c.) functions to approach our FBSVI. L^p -estimates for the solution of the penalized FBSDE on the whole interval are necessary (see the proof of our Proposition 21 and 22). However, the method in Cvitanić and Ma [9] can only give L^2 -estimates. Consequently, we should adapt the induction method introduced by Delarue in [10] to obtain the L^p -estimates in our framework (see Proposition 20).

Moreover, we will prove that the function u defined through the solution of our FBSVI (see (60)) is a viscosity solution of our new kind of quasilinear PVI. But because of the existence of the subdifferential operators, the continuity of u is not obvious at all. Therefore, we give a detailed proof in Proposition 24. For this, we separate the proof into two steps: To prove that u is right continuous w.r.t. t and continuous w.r.t. x in step 1, as well as left continuous w.r.t. t and continuous w.r.t. x in step 2.

The paper is organized as follows: In Section 2 we formulate the problem and we give the

definition of the viscosity solution of our new kind of PVI. Section 3 is devoted to prove the uniqueness of the viscosity solutions of PVI. In order to show the existence of the viscosity solution, in Section 4, we study general FBSVIs. More precisely, we obtain the existence and uniqueness of the solutions of FBSVIs. To do this, in subsection 4.1, we give the assumptions. In subsection 4.2, we introduce the penalized FBSDEs which are got by Yosida approximation for the subdifferential operators of the FBSVIs. Moreover, a priori estimates are established. Subsection 4.3 is devoted to the uniform L^p -estimates of the penalized FBSDEs. For this, we establish first the L^p -estimates on a small time interval and then we extend them to the whole interval by adapting the method developed by Delarue [10]. In subsection 4.4, based on the classical estimates for solving forward SVIs and BSVIs (see Proposition 21 and 22), we prove the existence and the uniqueness of the solution of FBSVIs. In Section 5, we prove that the function u defined as in (60) is the viscosity solution of such PVI. Finally, in the appendix, we provide a priori estimates for the penalized FBSDEs and some auxiliary results.

2 Formulation of the problem

We consider the following quasilinear PVI:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial s}(s, x) + (\mathcal{L}u)(s, x, u(s, x), (\nabla u(s, x))^* \sigma(s, x, u(s, x))) \\ \quad + f(s, x, u(s, x), (\nabla u(s, x))^* \sigma(s, x, u(s, x))) \in \partial \varphi(u(s, x)) + \langle \partial \psi(x), \nabla u(s, x) \rangle, \\ \quad (s, x) \in [0, T] \times \text{Dom} \psi, \\ u(T, x) = g(x), \quad x \in \text{Dom} \psi, \end{array} \right. \quad (1)$$

where the operator \mathcal{L} is defined by

$$(\mathcal{L}v)(s, x, y, z) := \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^*)_{i,j}(s, x, y) \frac{\partial^2 v}{\partial x_i \partial x_j}(s, x) + \sum_{i=1}^n b_i(s, x, y, z) \frac{\partial v}{\partial x_i}(s, x).$$

for $v \in C^{1,2}([0, T] \times \mathbb{R}^n)$. The functions $b : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times d}$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are jointly continuous. The operator $\partial \psi$ (resp. $\partial \varphi$) is the subdifferential of function ψ (resp. φ) which satisfies:

- (H'_1) The function $\psi : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ is convex l.s.c. with $0 \in \text{Int}(\text{Dom} \psi)$ and $\psi(z) \geq \psi(0) = 0$, for all $z \in \mathbb{R}^n$.
- (H'_2) The function $\varphi : \mathbb{R} \rightarrow (-\infty, +\infty]$ is convex l.s.c. such that $\varphi(y) \geq \varphi(0) = 0$ for all $y \in \mathbb{R}$.

We put

$$\begin{aligned} \text{Dom} \psi &= \{u \in \mathbb{R}^n : \psi(u) < \infty\}, \\ \partial \psi(u) &= \{u^* \in \mathbb{R}^n : \langle u^*, v - u \rangle + \psi(u) \leq \varphi(v), \ v \in \mathbb{R}^n\}, \\ \text{Dom}(\partial \psi) &= \{u \in \mathbb{R}^n : \partial \psi(u) \neq \emptyset\}, \end{aligned}$$

and we write $(u, u^*) \in \partial \psi$ if $u \in \text{Dom}(\partial \psi)$ and $u^* \in \partial \psi(u)$. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n .

We also mention that the multivalued subdifferential operator $\partial\psi$ is a monotone operator, i.e. $\langle u^* - v^*, u - v \rangle \geq 0$, for all $(u, u^*), (v, v^*) \in \partial\psi$.

Now we give the definition of a viscosity solution of PVI (1) in the language of sub- and super-jets:

Definition 1 *Let $u \in C([0, T] \times \text{Dom}\psi)$ satisfies $u(T, x) = g(x)$, $x \in \text{Dom}\psi$. The function u is called a viscosity subsolution (resp. supersolution) of PVI (1), if for all $(t, x) \in [0, T] \times \text{Dom}\psi$, $u(t, x) \in \text{Dom}\varphi$ and for any $(p, q, X) \in \mathcal{P}^{2,+}u(t, x)^*$ (resp. $(p, q, X) \in \mathcal{P}^{2,-}u(t, x)$),*

$$\begin{aligned} -p - \frac{1}{2}\text{Tr}(\sigma\sigma^*(t, x, u(t, x))X) - \langle b(t, x, u(t, x), q^*\sigma(t, x, u(t, x))), q \rangle \\ - f(t, x, u(t, x), q^*\sigma(t, x, u(t, x))) \leq -\varphi'_-(u(t, x)) - \partial\psi_*(x, q) \end{aligned} \quad (2)$$

(resp.

$$\begin{aligned} -p - \frac{1}{2}\text{Tr}(\sigma\sigma^*(t, x, u(t, x))X) - \langle b(t, x, u(t, x), q^*\sigma(t, x, u(t, x))), q \rangle \\ - f(t, x, u(t, x), q^*\sigma(t, x, u(t, x))) \geq -\varphi'_+(u(t, x)) - \partial\psi^*(x, q). \end{aligned} \quad (3)$$

Here $\varphi'_-(y)$ (resp. $\varphi'_+(y)$) denotes the left (resp. right) derivative of φ at point y , and

$$\partial\psi_*(x, q) := \liminf_{(x', q') \rightarrow (x, q), x^* \in \partial\psi(x')} \langle x^*, q' \rangle, \quad (x, q) \in \text{Dom}\psi \times \mathbb{R}^d,$$

and $\partial\psi^*(x, q) := -\partial\psi_*(x, -q)$ (for $\partial\psi_*$ and $\partial\psi^*$, see also [26]).

The function u is called a viscosity solution of PVI (1) if it is both a viscosity sub- and super-solution.

Remark 2 *Using the definition of $\partial\psi_*(x, q)$, and the fact that $y \mapsto \varphi'_-(y)$ is left continuous in $\text{Int}(\text{Dom}\varphi) \subset \mathbb{R}$ and increasing in $\text{Dom}\varphi$, we see that (2) is not only satisfied for $(p, q, X) \in \mathcal{P}^{2,+}u(t, x)$, but also for all $(p, q, X) \in \overline{\mathcal{P}}^{2,+}u(t, x)$. Similarly, (3) holds also for $(p, q, X) \in \overline{\mathcal{P}}^{2,-}u(t, x)$.*

We shall also use the following equivalent definition of a viscosity solution.

Definition 3 *Let $u \in C([0, T] \times \text{Dom}\psi)$ satisfies $u(T, x) = g(x)$, $x \in \text{Dom}\psi$. The function u is called a viscosity subsolution (resp. supersolution) of PVI (1), if for all $(t, x) \in [0, T] \times \text{Dom}\psi$, $u(t, x) \in \text{Dom}(\varphi)$, and if whenever $\Phi \in C^{1,2}([0, T] \times \text{Dom}\psi)$ and $(t, x) \in [0, T] \times \text{Dom}\psi$ is a local maximum (resp. minimum) point of $u - \Phi$, we have*

$$\begin{aligned} \frac{\partial\Phi}{\partial s}(t, x) + (\mathcal{L}u)(t, x, u(t, x), (\nabla\Phi(t, x))^*\sigma(t, x, u(t, x))) \\ + f(t, x, u(t, x), (\nabla\Phi(t, x))^*\sigma(t, x, u(t, x))) \geq \varphi'_-(u(t, x)) + \partial\psi_*(x, \nabla\Phi(t, x)) \end{aligned}$$

*Let $u \in C([0, T] \times \text{Dom}\psi)$ and $(t, x) \in [0, T] \times \text{Dom}\psi$. Denote by $\mathcal{P}^{2,+}u(t, x)$ the set of triples $(p, q, X) \in \mathbb{R} \times \mathbb{R}^n \times S(n)$ ($S(n)$ denotes the set of all $n \times n$ symmetric nonnegative matrices), such that

$$u(s, y) \leq u(t, x) + p(s - t) + \langle q, y - x \rangle + \frac{1}{2}\langle X(y - x), y - x \rangle + o(|s - t| + |y - x|^2), \quad (s, y) \rightarrow (t, x)$$

we call $\mathcal{P}^{2,+}u(t, x)$ the parabolic super-jet of u at (t, x) . Similarly, we define the parabolic sub-jet of u at (t, x) , denoted by $\mathcal{P}^{2,-}u(t, x)$ as the set of triples $(p, q, X) \in \mathbb{R} \times \mathbb{R}^n \times S(n)$ such that

$$u(s, y) \geq u(t, x) + p(s - t) + \langle q, y - x \rangle + \frac{1}{2}\langle X(y - x), y - x \rangle + o(|s - t| + |y - x|^2), \quad (s, y) \rightarrow (t, x).$$

(resp.

$$\begin{aligned} & \frac{\partial \Phi}{\partial s}(t, x) + (\mathcal{L}u)(t, x, u(t, x), (\nabla \Phi(t, x))^* \sigma(t, x, u(t, x))) \\ & + f(t, x, u(t, x), (\nabla \Phi(t, x))^* \sigma(t, x, u(t, x))) \leq \varphi'_+(u(t, x)) + \partial \psi^*(x, \nabla \Phi(t, x)). \end{aligned}$$

Finally, the function u is called a viscosity solution of PVI (1) if it is both a viscosity sub- and super-solution.

3 Uniqueness of viscosity solutions of PVIs

In this section, we prove the uniqueness of the viscosity solution of PVI (1) by extending and adapting the approaches of Barles, Buckdahn and Pardoux [4] and Cvitanić and Ma [9].

Theorem 4 *We assume that $\text{Dom}\psi$ is locally compact, b, σ, f, g are all jointly continuous. Moreover, suppose that b, f are Lipschitz continuous w.r.t. (x, y, z) and $\sigma(t, x, y)$ does not depend on y as well as Lipschitz continuous w.r.t. x . Then under the assumptions of (H'_1) and (H'_2) , PVI (1) has at most one viscosity solution in the class of functions which are Lipschitz continuous in x uniformly w.r.t. t and continuous in t .*

Remark 5 *We mention that it is sufficient to show that if u is a subsolution and v is a supersolution such that $u(T, x) = g(x) = v(T, x)$, $x \in \text{Dom}\psi$ and u, v are Lipschitz continuous in x uniformly w.r.t. t and continuous in t , then $u \leq v$ for all $(t, x) \in [0, T] \times \text{Dom}\psi$.*

Since u and v are continuous, we only need to show that $u \leq v$ for all $(t, x) \in (0, T) \times \text{Int}(\text{Dom}\psi)$. Let us define for each $r > 0$ a subset of $\text{Dom}\psi$

$$(\text{Dom}\psi)_r := \{x \in \text{Dom}\psi \mid d(x, \partial \text{Dom}\psi) \geq r\},$$

where $d(x, \partial \text{Dom}\psi) = \inf_{x' \in \partial \text{Dom}\psi} |x - x'|$. Let us choose $r_0 > 0$ such that $(\text{Dom}\psi)_{r_0} \neq \emptyset$ for all $0 < r \leq r_0$. Then it suffices to show that for every $0 < r \leq r_0$, we have $u \leq v$ on $(t, x) \in (0, T) \times (\text{Dom}\psi)_r$.

Before proving the Theorem 4, we recall the following lemma:

Lemma 6 (Lemma 2.3 [26]) *Let us denote by $\overline{\text{Dom}\psi}$ the closure of $\text{Dom}\psi$. We have $\partial \psi_*(x, q) = \inf_{x^* \in \partial \psi(x)} \langle x^*, q \rangle$, for $(x, q) \in \text{Int}(\text{Dom}\psi) \times \mathbb{R}^n$ or for $(x, q) \in \partial(\text{Dom}\psi) \times \mathbb{R}^n$ with $\inf_{n \in N_{\overline{\text{Dom}\psi}}(x)} \langle n, q \rangle > 0$. Here*

$$N_{\overline{\text{Dom}\psi}}(x) := \left\{ x_* \in \mathbb{R}^n \mid |x_*|^2 = 1, \langle x_*, z - x \rangle \leq 0, \text{ for all } z \in \overline{\text{Dom}\psi} \right\}.$$

In addition, we need the following lemma generalizing Lemma 7.2 [9].

Lemma 7 *Suppose that the assumptions of Theorem 4 are satisfied and u is a subsolution and v is a supersolution of (1) such that $u(T, x) = g(x) = v(T, x)$, $x \in \text{Dom}\psi$. Moreover, we assume that u, v are Lipschitz continuous in x , uniformly w.r.t. t , and continuous in t .*

Then for all $0 < r \leq r_0$, the function $\omega := u - v$ is a viscosity subsolution of the following equation:

$$\begin{cases} \min\{\omega, F_{u,v}(t, x, \omega, \omega_t, D\omega, D^2\omega) + \partial\psi_*(x, D\omega)\} = 0, & (t, x) \in [0, T] \times (Dom\psi)_r, \\ \omega(T, x) = 0, & x \in (Dom\psi)_r, \end{cases} \quad (4)$$

where

$$F_{u,v}(t, x, r, p, q, X) := -p - \frac{1}{2}Tr\{\sigma\sigma^*(t, x)X\} - \langle b(t, x, u(t, x), 0), q \rangle - \tilde{K}[|r| + |q| \cdot |\sigma(t, x)|]. \quad (5)$$

Here $\tilde{K} > 0$ is a constant depending only on the Lipschitz constants of the function b, f, u, v .

Proof. Fix $r \in (0, r_0)$ and $(t_0, x_0) \in [0, T] \times (Dom\psi)_r$. Let $\Phi \in C^{1,2}([0, T] \times Dom\psi)$ be such that (t_0, x_0) is a maximal point of $\omega - \Phi$ and $\omega(t_0, x_0) = \Phi(t_0, x_0)$. We assume w.l.o.g. that (t_0, x_0) is a strict, global maximum point of $\omega - \Phi$. Moreover, the Lipschitz property of u and v allows to assume that $D\Phi$ is uniformly bounded: $|D\Phi| \leq K_{u,v}$. We are going to prove that

$$\min\{\omega(t_0, x_0), F_{u,v}(t_0, x_0, \omega, \Phi_t, D\Phi, D^2\Phi) + \partial\psi_*(x_0, D\Phi)\} \leq 0. \quad (6)$$

Now for arbitrarily given $\alpha > 0$, we define

$$\Psi_\alpha(t, x, y) := u(t, x) - v(t, y) - \frac{\alpha}{2}|x - y|^2 - \Phi(t, x).$$

We choose $R > 0$ sufficiently large such that $(t_0, x_0) \in (Dom\psi)_{r,R}$, where $(Dom\psi)_{r,R} := \{x \in (Dom\psi)_r \mid |x| \leq R\}$. Then there exists $(t_\alpha, x_\alpha, y_\alpha) \in [0, T] \times (Dom\psi)_{r,R}^2$ such that $\Psi_\alpha(t_\alpha, x_\alpha, y_\alpha) = \max_{[0,T] \times (Dom\psi)_{r,R}^2} \Psi_\alpha(t, x, y)$.

From Proposition 3.7 [7] we have

$$\begin{cases} (i) & (t_\alpha, x_\alpha, y_\alpha) \rightarrow (t_0, x_0, x_0), \text{ as } \alpha \rightarrow \infty; \\ (ii) & \alpha|x_\alpha - y_\alpha|^2 \text{ is bounded and tends to zero, as } \alpha \rightarrow \infty. \end{cases}$$

Moreover, from $\omega(t_\alpha, x_\alpha) - \Phi(t_\alpha, x_\alpha) \leq \omega(t_0, x_0) - \Phi(t_0, x_0) = 0$ we have

$$\begin{aligned} 0 &= \omega(t_0, x_0) - \Phi(t_0, x_0) = \Psi_\alpha(t_0, x_0, x_0) \leq \Psi_\alpha(t_\alpha, x_\alpha, y_\alpha) \\ &= \omega(t_\alpha, x_\alpha) - \Phi(t_\alpha, x_\alpha) + v(t_\alpha, x_\alpha) - v(t_\alpha, y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \\ &\leq v(t_\alpha, x_\alpha) - v(t_\alpha, y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2, \end{aligned}$$

from where we deduce that $\alpha|x_\alpha - y_\alpha| \leq 2 \left| \frac{v(t_\alpha, x_\alpha) - v(t_\alpha, y_\alpha)}{x_\alpha - y_\alpha} \right| \leq 2K_v$, where K_v is the Lipschitz constant of v w.r.t. x .

We can assume that $u(t_0, x_0) > v(t_0, x_0)$; if not, (6) holds obviously. Thus, from the continuity of u and v , it follows that, for some $\alpha_0 > 0$, we have $u(t_\alpha, x_\alpha) > v(t_\alpha, y_\alpha)$ for $\alpha \geq \alpha_0$.

Let $\alpha > \alpha_0$. From Theorem 8.3 [7], we obtain that, for any $\delta > 0$, there exists $(X^\delta, Y^\delta) \in S(n) \times S(n)$ and $c^\delta \in \mathbb{R}^n$ such that

$$\begin{aligned} (c^\delta + \frac{\partial\Phi}{\partial t}(t_\alpha, x_\alpha), \alpha(x_\alpha - y_\alpha) + D\Phi(x_\alpha, y_\alpha), X^\delta) &\in \bar{\mathcal{P}}^{2,+}u(t_\alpha, x_\alpha); \\ (c^\delta, \alpha(x_\alpha - y_\alpha), Y^\delta) &\in \bar{\mathcal{P}}^{2,-}v(t_\alpha, y_\alpha) \end{aligned}$$

and

$$\begin{pmatrix} X^\delta & 0 \\ 0 & -Y^\delta \end{pmatrix} \leq A + \delta A^2,$$

where $A = \begin{pmatrix} D^2\Phi(t_\alpha, x_\alpha) + \alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix}$.

From Definition 1 and Remark 2 it follows that

$$\begin{aligned} & c^\delta + \frac{\partial\Phi}{\partial t}(t_\alpha, x_\alpha) + \frac{1}{2}Tr(\sigma\sigma^*(t_\alpha, x_\alpha)X^\delta) \\ & + \langle b(t_\alpha, x_\alpha, u(t_\alpha, x_\alpha), [\alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha)]^*\sigma(t_\alpha, x_\alpha)), \alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha) \rangle \\ & + f(t_\alpha, x_\alpha, u(t_\alpha, x_\alpha), [\alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha)]^*\sigma(t_\alpha, x_\alpha)) \\ & \geq \varphi'_-(u(t_\alpha, x_\alpha)) + \partial\psi_*(x_\alpha, \alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha)), \end{aligned}$$

and

$$\begin{aligned} & c^\delta + \frac{1}{2}Tr(\sigma\sigma^*(t_\alpha, y_\alpha)Y^\delta) + \langle b(t_\alpha, y_\alpha, v(t_\alpha, y_\alpha), [\alpha(x_\alpha - y_\alpha)]^*\sigma(t_\alpha, y_\alpha)), \alpha(x_\alpha - y_\alpha) \rangle \\ & + f(t_\alpha, y_\alpha, v(t_\alpha, y_\alpha), [\alpha(x_\alpha - y_\alpha)]^*\sigma(t_\alpha, y_\alpha)) \leq \varphi'_+(v(t_\alpha, y_\alpha)) + \partial\psi^*(y_\alpha, \alpha(x_\alpha - y_\alpha)). \end{aligned}$$

Then we have

$$\begin{aligned} & \varphi'_+(v(t_\alpha, y_\alpha)) - \varphi'_-(u(t_\alpha, x_\alpha)) \\ & + \partial\psi^*(y_\alpha, \alpha(x_\alpha - y_\alpha)) - \partial\psi_*(x_\alpha, \alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha)) \geq -\frac{\partial\Phi}{\partial t}(t_\alpha, x_\alpha) \\ & - \left\{ \frac{1}{2}Tr(\sigma\sigma^*(t_\alpha, x_\alpha)X^\delta) - \frac{1}{2}Tr(\sigma\sigma^*(t_\alpha, y_\alpha)Y^\delta) \right\} \quad (=:-I_1^{\alpha,\delta}) \\ & - \left\{ \langle b(t_\alpha, x_\alpha, u(t_\alpha, x_\alpha), [\alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha)]^*\sigma(t_\alpha, x_\alpha)), \alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha) \rangle \right. \\ & \quad \left. - b(t_\alpha, y_\alpha, v(t_\alpha, y_\alpha), [\alpha(x_\alpha - y_\alpha)]^*\sigma(t_\alpha, y_\alpha)), \alpha(x_\alpha - y_\alpha) \rangle \right\} \quad (=:-I_2^{\alpha,\delta}) \\ & - \left\{ f(t_\alpha, x_\alpha, u(t_\alpha, x_\alpha), [\alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha)]^*\sigma(t_\alpha, x_\alpha)) \right. \\ & \quad \left. - f(t_\alpha, y_\alpha, v(t_\alpha, y_\alpha), [\alpha(x_\alpha - y_\alpha)]^*\sigma(t_\alpha, y_\alpha)) \right\} \quad (=:-I_3^{\alpha,\delta}). \end{aligned} \tag{7}$$

We observe that the same argument as in Lemma 7.2 [9] yields

$$\begin{aligned} & \lim_{\alpha \rightarrow \infty} \lim_{\delta \rightarrow 0} (I_1^{\alpha,\delta} + I_2^{\alpha,\delta} + I_3^{\alpha,\delta}) \\ & \leq \frac{1}{2}Tr(\sigma\sigma^*(t_0, x_0)D^2\Phi(t_0, x_0)) + \langle b(t_0, x_0, u(t_0, x_0), 0), D\Phi(t_0, x_0) \rangle \\ & + \tilde{K} \{ |u(t_0, x_0) - v(t_0, y_0)| + |D\Phi(t_0, x_0)| \cdot |\sigma(t_0, x_0)| \}. \end{aligned} \tag{8}$$

Let us calculate now the left-hand side of (7). Since $u(t_\alpha, x_\alpha) > v(t_\alpha, y_\alpha)$, we have

$$\varphi'_+(v(t_\alpha, y_\alpha)) - \varphi'_-(u(t_\alpha, x_\alpha)) \leq 0. \tag{9}$$

Moreover, since ψ is convex, $x_\alpha, y_\alpha \in \text{Int}(\text{Dom}\psi)$ and $\langle y^*, \alpha(x_\alpha - y_\alpha) \rangle - \langle x^*, \alpha(x_\alpha - y_\alpha) \rangle \leq 0$, for $x^* \in \partial\psi(x_\alpha)$, $y^* \in \partial\psi(y_\alpha)$. By using Lemma 6, it follows that

$$\begin{aligned}
& \partial\psi^*(y_\alpha, \alpha(x_\alpha - y_\alpha)) - \partial\psi_*(x_\alpha, \alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha)) \\
&= \sup_{y^* \in \partial\psi(y_\alpha)} \langle y^*, \alpha(x_\alpha - y_\alpha) \rangle - \inf_{x^* \in \partial\psi(x_\alpha)} \langle x^*, \alpha(x_\alpha - y_\alpha) + D\Phi(t_\alpha, x_\alpha) \rangle \\
&\leq \sup_{y^* \in \partial\psi(y_\alpha)} \langle y^*, \alpha(x_\alpha - y_\alpha) \rangle - \inf_{x^* \in \partial\psi(x_\alpha)} \langle x^*, \alpha(x_\alpha - y_\alpha) \rangle \\
&\quad - \inf_{x^* \in \partial\psi(x_\alpha)} \langle x^*, D\Phi(t_\alpha, x_\alpha) \rangle \\
&\leq - \inf_{x^* \in \partial\psi(x_\alpha)} \langle x^*, D\Phi(t_\alpha, x_\alpha) \rangle \\
&= -\partial\psi_*(x_\alpha, D\Phi(t_\alpha, x_\alpha)).
\end{aligned} \tag{10}$$

Finally, letting $\delta \rightarrow 0$ and after letting $\alpha \rightarrow \infty$ in (7), by considering (8)-(10), we obtain

$$\begin{aligned}
& -\frac{\partial\Phi}{\partial t}(t_0, x_0) - \frac{1}{2}Tr(\sigma\sigma^*(t_0, x_0)D^2\Phi(t_0, x_0)) - \langle b(t_0, x_0, u(t_0, x_0), 0), D\Phi(t_0, x_0) \rangle \\
& -\tilde{K} \{|u(t_0, x_0) - v(t_0, y_0)| + |D\Phi(t_0, x_0)| \cdot |\sigma(t_0, x_0)|\} \leq -\partial\psi_*(x_0, D\Phi(t_0, x_0)).
\end{aligned}$$

Therefore, from (5),

$$F_{u,v}(t_0, x_0, \omega, \frac{\partial\Phi}{\partial t}, D\Phi, D^2\Phi) + \partial\psi_*(x_0, D\Phi) \leq 0,$$

evaluated at (t_0, x_0) . ■

Using the approach in Lemma 3.8 [4], we construct a suitable supersolution for (4).

Lemma 8 *For any $\tilde{A} > 0$, there exists $\tilde{C} > 0$ such that the function*

$$\chi(t, x) = \exp \left\{ [\tilde{C}(T - t) + \tilde{A}]\eta(x) \right\},$$

with

$$\eta(x) = \left\{ \log \left[(|x|^2 + 1)^{1/2} \right] + 1 \right\}^2,$$

satisfies

$$\min \left\{ \chi(t, x), F_{u,v}(t, x, \chi, \frac{\partial\chi}{\partial t}, D\chi, D^2\chi) + \partial\psi_*(x, D\chi) \right\} > 0, \quad \text{in } [t_1, T] \times (\text{Dom}\psi)_r,$$

where $t_1 = (T - \tilde{A}/\tilde{C})^+$.

Proof. A straightforward computation yields

$$\begin{aligned}
D\chi(t, x) &= \left[\tilde{C}(T - t) + \tilde{A} \right] \chi(t, x) D\eta(x) = \left[\tilde{C}(T - t) + \tilde{A} \right] \chi(t, x) \frac{2[\eta(x)]^{1/2}}{1 + |x|^2} x, \\
|D\chi(t, x)| &\leq 4\tilde{A} \frac{[\eta(x)]^{1/2}}{[1 + |x|^2]^{1/2}} \chi(t, x), \quad |D^2\chi(t, x)| \leq 16(\tilde{A}^2 + \tilde{A}) \frac{\eta(x)}{[1 + |x|^2]} \chi(t, x).
\end{aligned}$$

Since $x \in (\text{Dom}\psi)_r \subset \text{Int}(\text{Dom}\psi)$ and $\langle x^*, x \rangle \geq 0$ for $x^* \in \partial\psi(x)$, it follows that

$$\partial\psi_*(x, D\chi) = \inf_{x^* \in \partial\psi(x)} \langle x^*, D\chi(t, x) \rangle = \left[\tilde{C}(T - t) + \tilde{A} \right] \chi(t, x) \frac{2[\eta(x)]^{1/2}}{1 + |x|^2} \inf_{x^* \in \partial\psi(x)} \langle x^*, x \rangle \geq 0.$$

Since b, σ grow at most linearly at infinity and u is Lipschitz in x , uniformly w.r.t. t , we have, evaluating at (t, x) ,

$$\begin{aligned} F_{u,v}(t, x, \chi, \frac{\partial \chi}{\partial t}, D\chi, D^2\chi) + \partial\psi_*(x, D\chi) &\geq F_{u,v}(t, x, \chi, \frac{\partial \chi}{\partial t}, D\chi, D^2\chi) \\ &= -\frac{\partial \chi}{\partial t} - \frac{1}{2}Tr\{\sigma\sigma^*(t, x)D^2\chi\} - \langle b(t, x, u(t, x), 0), D\chi \rangle - \tilde{K}[|\chi| + |D\chi| \cdot |\sigma(t, x)|] \\ &\geq \chi(t, x) \left[\tilde{C}\eta(x) - 16(\tilde{A}^2 + \tilde{A})C\eta(x) - 4\tilde{A}C[\eta(x)]^{1/2} - \tilde{K} - 4\tilde{A}\tilde{K}C[\eta(x)]^{1/2} \right], \end{aligned}$$

where C is a constant independent of \tilde{C} . Since $\eta(x) \geq 1$, we can choose \tilde{C} large enough such that the quantity in the brackets is strictly positive. Consequently, taking into account that $\chi(t, x) > 0$, we can conclude:

$$\min\{\chi(t, x), F_{u,v}(t, x, \chi, \frac{\partial \chi}{\partial t}, D\chi, D^2\chi) + \partial\psi_*(x, D\chi)\} > 0, \quad \text{in } [t_1, T] \times (Dom\psi)_r.$$

■

Proof of Theorem 4. We only need to show that $\omega \leq 0$ on $[0, T] \times (Dom\psi)_r$ for any $r \in (0, r_0]$. Now let us choose \tilde{A} and \tilde{C} as in Lemma 8. Recalling that $\omega = u - v$ is Lipschitz in x , uniformly w.r.t. t , we remark that

$$\lim_{|x| \rightarrow \infty} |\omega(t, x)| \exp \left\{ -\tilde{A} \left[\log \left((|x|^2 + 1)^{1/2} \right) \right]^2 \right\} = 0,$$

uniformly w.r.t. $t \in [0, T]$. This implies that, for all $\varepsilon > 0$, $[\omega(t, x) - \varepsilon\chi(t, x)] e^{-\tilde{K}(T-t)}$ is bounded from above in $[t_1, T] \times (Dom\psi)_r$, with \tilde{K} as in (5). Now we define

$$M_\varepsilon^r := \max_{[t_1, T] \times (Dom\psi)_r} [\omega(t, x) - \varepsilon\chi(t, x)] e^{-\tilde{K}(T-t)},$$

and we suppose that the maximum M_ε^r is achieved at some point (t_0, x_0) . We claim that $M_\varepsilon^r \leq 0$ for all $r, \varepsilon > 0$. This holds obviously true if $t_0 = T$. Indeed, $M_\varepsilon^r = -\varepsilon\chi(T, x) \leq 0$. Thus, we can assume that $t_0 \in [0, T)$. Now we suppose that $M_\varepsilon^r > 0$ for some $r, \varepsilon > 0$, we will construct a contradiction. In fact, if we define

$$\Phi(t, x) := \varepsilon\chi(t, x) + M_\varepsilon^r e^{\tilde{K}(T-t)} = \varepsilon\chi(t, x) + [\omega(t_0, x_0) - \varepsilon\chi(t_0, x_0)] e^{-\tilde{K}(t-t_0)},$$

we have $\Phi \in C^{1,2}([0, T] \times (Dom\psi)_r)$, $\Phi(t_0, x_0) = \omega(t_0, x_0)$ and $\omega(t, x) - \Phi(t, x) \leq 0$, for all $(t, x) \in [0, T] \times (Dom\psi)_r$. From $M_\varepsilon^r > 0$, we have $\omega(t_0, x_0) > \varepsilon\chi(t_0, x_0) > 0$. Then, from Lemma 7 it follows that

$$F_{u,v}(t_0, x_0, \Phi, \frac{\partial \Phi}{\partial t}, D\Phi, D^2\Phi) + \partial\psi_*(x_0, D\Phi) \leq 0.$$

Moreover, since, at (t_0, x_0) ,

$$\begin{aligned}
0 &\geq F_{u,v}(t_0, x_0, \Phi, \frac{\partial \Phi}{\partial t}, D\Phi, D^2\Phi) + \partial\psi_*(x_0, D\Phi) \\
&= -\frac{\partial \Phi}{\partial t} - \frac{1}{2}Tr\{\sigma\sigma^*(t_0, x_0)D^2\Phi\} - \langle b(t_0, x_0, u(t_0, x_0), 0), D\Phi \rangle \\
&\quad - \tilde{K}[|\Phi| + |D\Phi| \cdot |\sigma(t_0, x_0)|] + \partial\psi_*(x_0, D\Phi) \\
&= -\varepsilon\frac{\partial \chi}{\partial t} + \tilde{K}M_\varepsilon^r e^{\tilde{K}(T-t)} - \varepsilon\frac{1}{2}Tr\{\sigma\sigma^*(t_0, x_0)D^2\chi\} - \varepsilon\langle b(t_0, x_0, u(t_0, x_0), 0), D\chi \rangle \\
&\quad - \tilde{K}[\varepsilon|\chi| + M_\varepsilon^r e^{\tilde{K}(T-t)} + \varepsilon|D\chi| \cdot |\sigma(t_0, x_0)|] + \varepsilon\partial\psi_*(x_0, D\chi) \\
&= \varepsilon \left[F_{u,v}(t_0, x_0, \chi, \frac{\partial \chi}{\partial t}, D\chi, D^2\chi) + \partial\psi_*(x_0, D\chi) \right]
\end{aligned}$$

we have

$$F_{u,v}(t_0, x_0, \chi, \frac{\partial \chi}{\partial t}, D\chi, D^2\chi) + \partial\psi_*(x_0, D\chi) \leq 0,$$

which contradicts Lemma 8. Therefore, $M_\varepsilon^r \leq 0$ which implies that $\omega(t, x) \leq \varepsilon\chi(t, x)$, for all $(t, x) \in [t_1, T] \times (Dom\psi)_r$. Letting $\varepsilon \rightarrow 0$, we get $\omega(t, x) \leq 0$ for all $(t, x) \in [t_1, T] \times (Dom\psi)_r$. Applying successively the similar argument on the interval $[t_2, t_1]$, if necessary, where $t_2 = (t_1 - \tilde{A}/\tilde{C})^+$, and then if $t_2 > 0$, on $[t_3, t_2]$, where $t_3 = (t_2 - \tilde{A}/\tilde{C})^+$, etc., we obtain, finally,

$$\omega(t, x) \leq 0, \text{ for all } (t, x) \in [0, T] \times (Dom\psi)_r, \quad r \in (0, r_0].$$

■

4 FBSVIs

In this section, in order to prepare our existence result for PVI (1), we study one general kind of FBSVIs in order to give a probabilistic interpretation for the viscosity solution of PVI (1).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and endowed with an \mathbb{R}^d -valued standard Brownian motion $\{B_t\}_{t \geq 0}$. We denote by $\{\mathcal{F}_t\}_{t \geq 0}$ the filtration generated by the Brownian motion B and augmented by the class of P -null sets of \mathcal{F} .

We consider the following FBSVI:

$$\begin{cases} dX_t + \partial\psi(X_t)dt \ni b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t, Z_t)dB_t, \\ -dY_t + \partial\varphi(Y_t)dt \ni f(t, X_t, Y_t, Z_t)dt - Z_t dB_t, \quad t \in [0, T], \\ X_0 = x, \quad Y_T = g(X_T), \end{cases} \quad (11)$$

where the processes X, Y, Z take values in $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^{m \times d}$, respectively, and the functions b, σ, f and g satisfy standard assumptions which we will give later. The operator $\partial\psi$ (resp. $\partial\varphi$) is the subdifferential of the convex l.s.c. function $\psi : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ (resp. $\varphi : \mathbb{R}^m \rightarrow (-\infty, +\infty]$).

Let us introduce some spaces of processes, which will be needed in what follows.

Let $t \in [0, T]$, $k \geq 1$. We define $BV([t, T]; \mathbb{R}^k)$ as the space of the functions $u : [t, T] \rightarrow \mathbb{R}^k$ with finite total variation on $[t, T]$, denoted by $\uparrow u \uparrow_{[t, T]}$. Moreover we endow this space with the norm

$$\|u\|_{BV([t, T]; \mathbb{R}^k)} = |u(t)| + \uparrow u \uparrow_{[t, T]}.$$

We also introduce the space $L^1(\Omega; BV([t, T]; \mathbb{R}^k))$ of the stochastic processes $u : \Omega \times [t, T] \rightarrow \mathbb{R}^k$ such that

$$E\|u\|_{BV([t, T]; \mathbb{R}^k)} < \infty.$$

The space $M_k^p[t, T]$, $p \geq 2$, denotes the Hilbert space of \mathbb{R}^k -valued $\{\mathcal{F}_s\}$ -progressively measurable processes $\{u(s), s \in [t, T]\}$ such that

$$\|u\|_{M^p[t, T]} := \left(E \left(\int_t^T |u(s)|^2 ds \right)^{p/2} \right)^{1/p} < \infty.$$

When $p = 2$, we set $\|\cdot\|_{M[t, T]} := \|\cdot\|_{M^2[t, T]}$. For $\lambda \in \mathbb{R}$, we define an equivalent norm on $M_k^2[t, T]$:

$$\|u\|_{M_\lambda[t, T]} := \left(E \int_t^T e^{-\lambda s} |u(s)|^2 ds \right)^{1/2}.$$

Moreover, let $H[t, T]$ be the subset of $M_n^2[t, T]$ consisting of all the continuous processes. For $\beta > 0$ and $\lambda \in \mathbb{R}$, we denote its completion under the norm

$$\|u\|_{\lambda, \beta, [t, T]} := e^{-\lambda T} E|u(T)|^2 + \beta \|u\|_{M_\lambda[t, T]}$$

by $\bar{H}[t, T]$. The notation $\bar{H}[t, T]$ takes into account that the completion of $H[t, T]$ under the norm $\|\cdot\|_{\lambda, \beta, [t, T]}$ does not depend on λ and β .

Let $S_k^2[t, T]$ be the set of all continuous $\{\mathcal{F}_s\}$ -progressively measurable processes $\{u(s), s \in [t, T]\}$ which values in \mathbb{R}^k such that

$$\|u\|_{S[t, T]} := \left(E \sup_{t \leq s \leq T} |u(s)|^2 \right)^{1/2} < \infty.$$

We also introduce an equivalent norm on $S_k^2[t, T]$:

$$\|u\|_{S_\lambda[t, T]} := \left(E \sup_{t \leq s \leq T} e^{-\lambda s} |u(s)|^2 \right)^{1/2}.$$

In what follows, if $t = 0$, we simplify the notations by writing, for example: $M_k^2 := M_k^2[0, T]$, $H := H[0, T]$.

Let us give the following definition.

Definition 9 A quintuple (X, Y, Z, V, U) of processes is called an adapted solution of FBSVI (11), if the following conditions are satisfied:

- (a₁) $X \in S_n^2$, $Y \in S_m^2$, $Z \in M_{m \times d}^2$, $V \in S_n^2 \cap L^1(\Omega; BV([0, T]; \mathbb{R}^n))$, $V_0 = 0$, $U \in M_m^2$,
- (a₂) $X_t \in \text{Dom}\psi$, $d\mathbb{P} \otimes dt$ a.e. and $\psi(X) \in L^1(\Omega \times [0, T]; \mathbb{R})$,

$$\int_s^t \langle z - X_r, dV_r \rangle + \int_s^t \psi(X_r) dr \leq (t - s)\psi(z), \quad \text{for all } z \in \mathbb{R}^n, \quad 0 \leq s \leq t \leq T, \quad \text{a.s.}$$
- (a₃) $(Y_t, U_t) \in \partial\varphi$, $d\mathbb{P} \otimes dt$ a.e. on $\Omega \times [0, T]$,
- (a₄) $X_t + V_t = x + \int_0^t b(s, X_s, Y_s, Z_s) ds + \int_0^t \sigma(s, X_s, Y_s, Z_s) dB_s$,

$$Y_t + \int_t^T U_s ds = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad \text{a.s.}$$

4.1 Assumptions

Now we give the following standard assumptions:

- (H₁) Let $\psi : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be a convex l.s.c. function with $0 \in \text{Int}(\text{Dom}\psi)$ and $\psi(z) \geq \psi(0) = 0$, for all $z \in \mathbb{R}^n$. Moreover, the initial point x from (11) belongs to $\text{Dom}\psi$.
- (H₂) Let $\varphi : \mathbb{R}^m \rightarrow (-\infty, +\infty]$ be a convex l.s.c. function s.t. $\varphi(y) \geq \varphi(0) = 0$ for all $y \in \mathbb{R}^m$.
- (H₃) The coefficients b, σ and f are defined on $\Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, s.t. $b(\cdot, \cdot, x, y, z)$, $\sigma(\cdot, \cdot, x, y, z)$ and $f(\cdot, \cdot, x, y, z)$ are $\{\mathcal{F}_t\}$ -progressively measurable processes, for all fixed $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$. The coefficient g is defined on $\Omega \times \mathbb{R}^n$ and $g(\cdot, x)$ is \mathcal{F}_T -measurable, for all fixed $x \in \mathbb{R}^n$.
- (H₄) The mapping $y \mapsto f(\omega, t, x, y, z) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is continuous and there exists a constant $L \geq 0$ and $\eta \in M_1^2$, such that for all (ω, t, y) , $|f(\omega, t, 0, y, 0)| \leq \eta(\omega, t) + L|y|$ (which implies that $f(\cdot, \cdot, 0, 0, 0) \in M_m^2$). Moreover, $b(\cdot, \cdot, 0, 0, 0) \in M_n^2$, $\sigma(\cdot, \cdot, 0, 0, 0) \in M_{n \times d}^2$ and $E|g(\cdot, 0)|^2 < \infty$.
- (H₅) There exist positive constants K, k_1, k_2 and a constant $\gamma \in \mathbb{R}$, such that for all $t, x, x_1, x_2, y, y_1, y_2, z, z_1, z_2$, a.s.
 - (i) $|b(t, x_1, y_1, z_1) - b(t, x_2, y_2, z_2)| \leq K(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$,
 - (ii) $|\sigma(t, x_1, y_1, z_1) - \sigma(t, x_2, y_2, z_2)|^2 \leq K^2(|x_1 - x_2|^2 + |y_1 - y_2|^2) + k_1^2|z_1 - z_2|^2$,
 - (iii) $|g(x_1) - g(x_2)| \leq k_2|x_1 - x_2|$,
 - (iv) $|f(t, x_1, y, z_1) - f(t, x_2, y, z_2)| \leq K(|x_1 - x_2| + |z_1 - z_2|)$,
 - (v) $\langle f(t, x, y_1, z) - f(t, x, y_2, z), y_1 - y_2 \rangle \leq \gamma|y_1 - y_2|^2$.

Remark 10 (1) We shall also introduce the following conditions:

(H₄') The mapping $y \mapsto f(\omega, t, x, y, z) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is continuous and there exist constants $L \geq 0$, $\rho_0 > 0$ and a process $\eta \in M_1^{3+\rho_0}$, such that for all (ω, t, y) , $|f(\omega, t, 0, y, 0)| \leq \eta(\omega, t) + L|y|$ (which implies that $f(\cdot, \cdot, 0, 0, 0) \in M_m^{3+\rho_0}$). Moreover, $b(\cdot, \cdot, 0, 0, 0) \in M_n^{3+\rho_0}$, $\sigma(\cdot, \cdot, 0, 0, 0) \in M_{n \times d}^{3+\rho_0}$ and $E|g(\cdot, 0)|^{3+\rho_0} < \infty$.

(H₅') $k_1 = 0$, i.e., σ does not depend on z .

(2) For simplification, we put $b^0(s) := b(\cdot, s, 0, 0, 0)$, $\sigma^0(s) := \sigma(\cdot, s, 0, 0, 0)$, $f^0(s) := f(\cdot, s, 0, 0, 0)$ and $g^0 := g(\cdot, 0)$.

These assumptions will be completed by compatibility hypotheses which were introduced by Cvitanić and Ma [9]:

$$(C1) \quad 0 \leq k_1 k_2 < 1,$$

$$(C2) \quad \text{If } k_2 = 0 \text{ then there exists } \alpha \in (0, 1), \text{ s.t. } \mu(\alpha, T) K C_3 < \bar{\lambda}_1,$$

$$(C3) \quad \text{If } k_2 > 0 \text{ then there exists } \alpha \in (k_1^2 k_2^2, 1), \text{ s.t. } \mu(\alpha, T) k_2^2 < 1 \text{ and } \bar{\lambda}_1 \geq \frac{K C_3}{k_2^2}.$$

Here

$$\begin{aligned} \mu(\alpha, T) &:= K(C_1 + K)B(\bar{\lambda}_2, T) + \frac{A(\bar{\lambda}_2, T)}{\alpha}(K C_2 + k_1^2), \\ A(\lambda, t) &= e^{-(\lambda \wedge 0)t}, \quad B(\lambda, t) = \int_0^t e^{-\lambda s} ds, \quad t \in [0, T], \\ \bar{\lambda}_1 &= \lambda - K(2 + C_1^{-1} + C_2^{-1}) - K^2, \quad \bar{\lambda}_2 = -\lambda - 2\gamma - K(C_3^{-1} + C_4^{-1}), \\ &\text{for } \lambda \in \mathbb{R} \text{ and constants } C_1, C_2, C_3, C_4 > 0. \end{aligned} \tag{12}$$

Remark 11 We mention that in Cvitanić and Ma [9], $\bar{\lambda}_1 = \frac{K C_3}{k_2^2}$ in (C3). However, it turns out that $\bar{\lambda}_1 \geq \frac{K C_3}{k_2^2}$ is enough. See the proof of Proposition 29 and Theorem 30 in the appendix of our paper.

4.2 Penalized FBSDEs and a priori estimates

In this section, we give a priori estimates on penalized equations related with FBSVI (11), inspired by [2], [9] and [19].

We begin with recalling the Yosida approximation for our convex l.s.c. function φ :

$$\varphi_\varepsilon(u) := \inf \left\{ \frac{1}{2\varepsilon} |u - v|^2 + \varphi(v) : v \in \mathbb{R}^m \right\}, \quad \varepsilon > 0, \quad u \in \mathbb{R}^m.$$

It is well known that the function φ_ε is convex and belongs to the class $C^1(\mathbb{R}^m)$. The gradient $\nabla \varphi_\varepsilon$ is a Lipschitz function with Lipschitz constant $1/\varepsilon$. We set

$$J_{\varepsilon, \varphi}(u) = u - \varepsilon \nabla \varphi_\varepsilon(u), \quad u \in \mathbb{R}^m.$$

The approximation φ_ε has the following properties (see [3] and [19]):

For all $u, v \in \mathbb{R}^m$, and $\varepsilon, \delta > 0$, we have

$$\begin{aligned}
(a) \quad & \varphi_\varepsilon(u) = \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon(u)|^2 + \varphi(J_{\varepsilon, \varphi}(u)), \\
(b) \quad & |J_{\varepsilon, \varphi}(u) - J_{\varepsilon, \varphi}(v)| \leq |u - v|, \\
(c) \quad & \nabla \varphi_\varepsilon(u) \in \partial \varphi(J_{\varepsilon, \varphi}(u)), \\
(d) \quad & 0 \leq \varphi_\varepsilon(u) \leq \langle \nabla \varphi_\varepsilon(u), u \rangle, \\
(e) \quad & \langle \nabla \varphi_\varepsilon(u) - \nabla \varphi_\delta(v), u - v \rangle \geq -(\varepsilon + \delta) |\nabla \varphi_\varepsilon(u)| |\nabla \varphi_\delta(v)|.
\end{aligned} \tag{13}$$

Moreover for our convex l.s.c. function ψ , we have in addition, the following property (see [2, 3]):

$$\begin{aligned}
(f) \quad & \text{For all } u_0 \in \text{Int}(\text{Dom} \psi), \text{ there exist } r_0 > 0, M_0 > 0, \text{ such that} \\
& r_0 |\nabla \psi_\varepsilon(x)| \leq \langle \nabla \psi_\varepsilon(x), x - u_0 \rangle + M_0, \text{ for all } \varepsilon > 0, x \in \mathbb{R}^n.
\end{aligned} \tag{14}$$

Let $\varepsilon > 0$. We consider the following penalized FBSDE using the Yosida approximation for ψ and φ :

$$\begin{cases} X_t^\varepsilon + \int_0^t \nabla \psi_\varepsilon(X_s^\varepsilon) ds = x + \int_0^t b(s, X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon) ds + \int_0^t \sigma(s, X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon) dB_s, \\ Y_t^\varepsilon + \int_t^T \nabla \varphi_\varepsilon(Y_s^\varepsilon) ds = g(X_T^\varepsilon) + \int_t^T f(s, X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon) ds - \int_t^T Z_s^\varepsilon dB_s, \end{cases} \quad t \in [0, T]. \tag{15}$$

From Theorem 30 (see Appendix), we have

Lemma 12 *Let the assumptions (H_1) – (H_5) be satisfied. We also assume $(C1)$ and either $(C2)$ or $(C3)$ hold for some $\lambda, \alpha, C_1, C_2, C_3$ and $C_4 = \frac{1-\alpha}{K}$. Then penalized FBSDE (15) has a unique adapted solution $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon) \in S_n^2 \times S_m^2 \times M_{n \times d}^2$. Moreover, if among the compatibility conditions, only $(C1)$ holds, then penalized FBSDE (15) has a unique adapted solution on $[0, T_0]$, but only for $T_0 > 0$ small enough.*

Remark 13 (1) *In our paper we only discuss the assumption $(C1)$ in combination with either $(C2)$ or $(C3)$. For the case that only $(C1)$ holds and $T_0 > 0$ small enough, all our results can be obtained similarly, so we omit it.*

(2) *As explained in Remark 3.2 [9], the compatibility conditions do not include the case of an arbitrary T since the constants introduced in (12) depend on T . However, under the condition $(C1)$, if $\gamma \leq -\Upsilon$, for some $\Upsilon > 0$ depending only on K, k_1, k_2 , then all our results holds for arbitrary T .*

Proposition 14 *Under the assumptions of Lemma 12, if $(X^{t,x}, Y^{t,x}, Z^{t,x}, V^{t,x}, U^{t,x})$ (resp. $(\tilde{X}^{t,\tilde{x}}, \tilde{Y}^{t,\tilde{x}}, \tilde{Z}^{t,\tilde{x}}, \tilde{V}^{t,\tilde{x}}, \tilde{U}^{t,\tilde{x}})$) is a solution of the FBSVI with initial time t and parameters (x, b, σ, f, g) (resp. $(\tilde{x}, \tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g})$), then there exists a constant C independent of (t, x, \tilde{x}) , such that*

$$E \left\{ \sup_{t \leq s \leq T} |X_s^{t,x} - \tilde{X}_s^{t,\tilde{x}}|^2 + \sup_{t \leq s \leq T} |Y_s^{t,x} - \tilde{Y}_s^{t,\tilde{x}}|^2 + \int_t^T |Z_s^{t,x} - \tilde{Z}_s^{t,\tilde{x}}|^2 ds \right\} \leq C \Delta_1, \tag{16}$$

where

$$\begin{aligned}\Delta_1 &= e^{-\lambda t}|x - \tilde{x}|^2 + E|g(X_T) - \tilde{g}(X_T)|^2 + E \int_t^T |b - \tilde{b}|^2(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds \\ &\quad + E \int_t^T |f - \tilde{f}|^2(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds + E \int_t^T |\sigma - \tilde{\sigma}|^2(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds.\end{aligned}$$

Proof. From Definition 9 (a_2), it follows that

$$\int_a^b \langle z - X_r^{t,x}, dV_r^{t,x} \rangle + \int_a^b \psi(X_r^{t,x}) dr \leq (b-a)\psi(z), \quad z \in \mathbb{R}^n, \quad t \leq a \leq b \leq T,$$

Recalling Proposition 1.2 of [2], we know that it is equivalent to

$$\int_a^b \langle y_r - X_r^{t,x}, dV_r^{t,x} \rangle + \int_a^b \psi(X_r^{t,x}) dr \leq \int_a^b \psi(y_r) dr, \quad y \in C([t, T]; \mathbb{R}^n), \quad t \leq a \leq b \leq T.$$

Consequently, we have

$$\begin{aligned}\int_a^b \langle \tilde{X}_r^{t,\tilde{x}} - X_r^{t,x}, dV_r^{t,x} \rangle + \int_a^b \psi(X_r) dr &\leq \int_a^b \psi(\tilde{X}_r^{t,\tilde{x}}) dr, \quad t \leq a \leq b \leq T, \\ \int_a^b \langle X_r^{t,x} - \tilde{X}_r^{t,\tilde{x}}, d\tilde{V}_r^{t,x} \rangle + \int_a^b \psi(\tilde{X}_r^{t,\tilde{x}}) dr &\leq \int_a^b \psi(X_r^{t,x}) dr, \quad t \leq a \leq b \leq T,\end{aligned}$$

which yields

$$\int_a^b \langle X_r^{t,x} - \tilde{X}_r^{t,\tilde{x}}, dV_r - d\tilde{V}_r^{t,x} \rangle \geq 0. \quad (17)$$

Moreover, from $(Y, U), (\tilde{Y}, \tilde{U}) \in \partial\varphi$, it follows

$$\langle Y - \tilde{Y}, U - \tilde{U} \rangle \geq 0. \quad (18)$$

Using (17) and (18), similarly to the estimates (80)-(82) of Proposition 29 (see the appendix), it follows

$$\begin{aligned}e^{-\lambda T} E|X_T^{t,x} - \tilde{X}_T^{t,\tilde{x}}|^2 + \bar{\lambda}_1^\delta \|X^{t,x} - \tilde{X}^{t,\tilde{x}}\|_{M_\lambda[t,T]}^2 &\leq K[C_1 + K(1+\delta)] \|Y^{t,x} - \tilde{Y}^{t,\tilde{x}}\|_{M_\lambda[t,T]}^2 \\ &\quad + [KC_2 + k_1^2(1+\delta)] \|Z^{t,x} - \tilde{Z}^{t,\tilde{x}}\|_{M_\lambda[t,T]}^2 + e^{-\lambda t}|x - \tilde{x}|^2 \\ &\quad + \frac{1}{\delta} \|(b - \tilde{b})(X^{t,x}, Y^{t,x}, Z^{t,x})\|_{M_\lambda[t,T]} + (1 + \frac{1}{\delta}) \|(\sigma - \tilde{\sigma})(X^{t,x}, Y^{t,x}, Z^{t,x})\|_{M_\lambda[t,T]},\end{aligned} \quad (19)$$

$$\begin{aligned}\|Y^{t,x} - \tilde{Y}^{t,\tilde{x}}\|_{M_\lambda[t,T]}^2 &\leq B(\bar{\lambda}_2^\delta, T) \left\{ k_2^2(1+\delta) e^{-\lambda T} E|X_T^{t,x} - \tilde{X}_T^{t,\tilde{x}}|^2 + KC_3 \|X^{t,x} - \tilde{X}^{t,\tilde{x}}\|_{M_\lambda[t,T]}^2 \right. \\ &\quad \left. + \frac{1}{\delta} \|(f - \tilde{f})(X^{t,x}, Y^{t,x}, Z^{t,x})\|_{M_\lambda[t,T]} + (1 + \frac{1}{\delta}) e^{-\lambda T} E|g(X_T^{t,x}) - \tilde{g}(X_T^{t,x})|^2 \right\},\end{aligned} \quad (20)$$

and

$$\begin{aligned}\|Z^{t,x} - \tilde{Z}^{t,\tilde{x}}\|_{M_\lambda[t,T]}^2 &\leq \frac{A(\bar{\lambda}_2^\delta, T)}{\alpha} \left\{ k_2^2(1+\delta) e^{-\lambda T} E|X_T^{t,x} - \tilde{X}_T^{t,\tilde{x}}|^2 + KC_3 \|X^{t,x} - \tilde{X}^{t,\tilde{x}}\|_{M_\lambda[t,T]}^2 \right. \\ &\quad \left. + \frac{1}{\delta} \|(f - \tilde{f})(X^{t,x}, Y^{t,x}, Z^{t,x})\|_{M_\lambda[t,T]} + (1 + \frac{1}{\delta}) e^{-\lambda T} E|g(X_T^{t,x}) - \tilde{g}(X_T^{t,x})|^2 \right\}.\end{aligned} \quad (21)$$

Then using the same argument as in Proposition 29, we obtain our results. \blacksquare

Remark 15 Putting $(X^{t,x}, Y^{t,x}, Z^{t,x}, V^{t,x}, U^{t,x}) = (0, 0, 0, 0, 0)$ which is the solution of FB-SVI with initial time t and parameters $(0, 0, 0, 0, 0)$, we see from Proposition 14 that there exists a constant C independent of (t, \tilde{x}) , such that

$$E \left\{ \sup_{t \leq s \leq T} |\tilde{X}_s^{t,\tilde{x}}|^2 + \sup_{t \leq s \leq T} |\tilde{Y}_s^{t,\tilde{x}}|^2 + \int_t^T |\tilde{Z}_s^{t,\tilde{x}}|^2 ds \right\} \leq C \Delta_2, \quad (22)$$

where

$$\Delta_2 = e^{-\lambda t} |\tilde{x}|^2 + E |\tilde{g}^0|^2 + E \int_t^T |\tilde{b}^0(s)|^2 ds + E \int_t^T |\tilde{f}^0(s)|^2 ds + E \int_t^T |\tilde{\sigma}^0(s)|^2 ds.$$

Proposition 16 Let the assumptions (H_1) – (H_5) be satisfied. We also assume $(C1)$ and either $(C2)$ or $(C3)$ hold for some $\lambda, \alpha, C_1, C_2, C_3$ and $C_4 = \frac{1-\alpha}{K}$. Then for all $\varepsilon_1, \varepsilon_2 > 0$,

$$\begin{aligned} & E \left\{ \sup_{0 \leq t \leq T} |X_t^{\varepsilon_1} - X_t^{\varepsilon_2}|^2 + \sup_{0 \leq t \leq T} |Y_t^{\varepsilon_1} - Y_t^{\varepsilon_2}|^2 + \int_0^T |Z_s^{\varepsilon_1} - Z_s^{\varepsilon_2}|^2 ds \right\} \\ & \leq C(\varepsilon_1 + \varepsilon_2) E \left[\int_0^T |\nabla \psi_{\varepsilon_1}(X_s^{\varepsilon_1})| |\nabla \psi_{\varepsilon_2}(X_s^{\varepsilon_2})| ds + \int_0^T |\nabla \varphi_{\varepsilon_1}(Y_s^{\varepsilon_1})| |\nabla \varphi_{\varepsilon_2}(Y_s^{\varepsilon_2})| ds \right], \end{aligned} \quad (23)$$

where C is a constant which does not depend on ε_1 nor on ε_2 .

Proof. We apply Itô's formula to $\left(e^{-\lambda s} e^{-\lambda'(t-s)} |X_s^{\varepsilon_1} - X_s^{\varepsilon_2}|^2 \right)_{0 \leq s \leq t}$. From $\langle \nabla \psi_{\varepsilon_1}(u) - \nabla \psi_{\varepsilon_2}(v), u - v \rangle \geq -(\varepsilon_1 + \varepsilon_2) |\nabla \psi_{\varepsilon_1}(u)| |\nabla \psi_{\varepsilon_2}(v)|$ (see (13)-(e)), similarly to Lemma 5.1 [9], it follows that

$$\begin{aligned} & e^{-\lambda T} E |X_T^{\varepsilon_1} - X_T^{\varepsilon_2}|^2 + \bar{\lambda}_1 \|X^{\varepsilon_1} - X^{\varepsilon_2}\|_{M_\lambda}^2 \leq K(C_1 + K) \|Y^{\varepsilon_1} - Y^{\varepsilon_2}\|_{M_\lambda}^2 \\ & + (KC_2 + k_1^2) \|Z^{\varepsilon_1} - Z^{\varepsilon_2}\|_{M_\lambda}^2 + 2(\varepsilon_1 + \varepsilon_2) E \int_0^T e^{-\lambda s} |\nabla \psi_{\varepsilon_1}(X_s^{\varepsilon_1})| |\nabla \psi_{\varepsilon_2}(X_s^{\varepsilon_2})| ds, \end{aligned} \quad (24)$$

where $\bar{\lambda}_1 = \lambda - K(2 + C_1^{-1} + C_2^{-1}) - K^2$ and C_1, C_2 are positive constants.

We apply again Itô's formula but now to $\left(e^{-\lambda s} e^{-\lambda'(s-t)} |Y_s^{\varepsilon_1} - Y_s^{\varepsilon_2}|^2 \right)_{t \leq s \leq T}$. Observing that $\langle \nabla \varphi_{\varepsilon_1}(u) - \nabla \varphi_{\varepsilon_2}(v), u - v \rangle \geq -(\varepsilon_1 + \varepsilon_2) |\nabla \varphi_{\varepsilon_1}(u)| |\nabla \varphi_{\varepsilon_2}(v)|$, we obtain

$$\begin{aligned} & \|Y^{\varepsilon_1} - Y^{\varepsilon_2}\|_{M_\lambda}^2 \leq B(\bar{\lambda}_2, T) \left[k_2^2 e^{-\lambda T} E |X_T^{\varepsilon_1} - X_T^{\varepsilon_2}|^2 + KC_3 \|Y^{\varepsilon_1} - Y^{\varepsilon_2}\|_{M_\lambda}^2 \right. \\ & \left. + 2(\varepsilon_1 + \varepsilon_2) E \int_0^T e^{-\lambda s} |\nabla \varphi_{\varepsilon_1}(Y_s^{\varepsilon_1})| |\nabla \varphi_{\varepsilon_2}(Y_s^{\varepsilon_2})| ds \right], \end{aligned} \quad (25)$$

$$\begin{aligned} & \|Z^{\varepsilon_1} - Z^{\varepsilon_2}\|_{M_\lambda}^2 \leq \frac{A(\bar{\lambda}_2, T)}{\alpha} \left[k_2^2 e^{-\lambda T} E |X_T^{\varepsilon_1} - X_T^{\varepsilon_2}|^2 + KC_3 \|Y^{\varepsilon_1} - Y^{\varepsilon_2}\|_{M_\lambda}^2 \right. \\ & \left. + 2(\varepsilon_1 + \varepsilon_2) E \int_0^T e^{-\lambda s} |\nabla \varphi_{\varepsilon_1}(Y_s^{\varepsilon_1})| |\nabla \varphi_{\varepsilon_2}(Y_s^{\varepsilon_2})| ds \right], \end{aligned} \quad (26)$$

where $\bar{\lambda}_2 = -\lambda - 2\gamma - K(C_3^{-1} + C_4^{-1})$ and C_3, C_4 are positive constants. Now from (24)-(26) as well as $(C1), (C2)$ or $(C1), (C3)$, by using Burkholder-Davis-Gundy (BDG) inequality, we check that there exists a constant C independent of ε_1 and ε_2 , such that (23) holds. \blacksquare

4.3 L^p -estimates for the penalized equations

We begin our study with the L^2 -estimates for penalized FBSDE (15).

Proposition 17 *Let the assumptions (H_1) – (H_5) be satisfied. We also assume $(C1)$ and either $(C2)$ or $(C3)$ hold for some $\lambda, \alpha, C_1, C_2, C_3$ and $C_4 = \frac{1-\alpha}{K}$. Then*

$$\begin{aligned} & \|X^{\varepsilon,t,x_1} - X^{\varepsilon,t,x_2}\|_{S[t,T]}^2 + \|Y^{\varepsilon,t,x_1} - Y^{\varepsilon,t,x_2}\|_{S[t,T]}^2 \\ & + \|Z^{\varepsilon,t,x_1} - Z^{\varepsilon,t,x_2}\|_{M[t,T]}^2 \leq C_T |x_1 - x_2|^2, \quad x_1, x_2 \in \mathbb{R}^n, \end{aligned} \quad (27)$$

with a constant C_T which is independent of (t, x_1, x_2) and ε .

Proof. We put $\hat{X}^\varepsilon = X^{\varepsilon,t,x_1} - X^{\varepsilon,t,x_2}$, $\hat{Y}^\varepsilon = Y^{\varepsilon,t,x_1} - Y^{\varepsilon,t,x_2}$ and $\hat{Z}^\varepsilon = Z^{\varepsilon,t,x_1} - Z^{\varepsilon,t,x_2}$. From (76)–(78), using $e^{-\lambda t} \leq e^{-(\lambda \wedge 0)t} \leq e^{-(\lambda \wedge 0)T}$, $A(\lambda, T-t) \leq A(\lambda, T)$ and $B(\lambda, T-t) \leq B(\lambda, T)$, we have the following estimates:

$$\begin{aligned} & e^{-\lambda T} E|\hat{X}_T^\varepsilon|^2 + \bar{\lambda}_1 \|\hat{X}^\varepsilon\|_{M_\lambda[t,T]}^2 \leq e^{-(\lambda \wedge 0)T} |x_1 - x_2|^2 \\ & + K(C_1 + K) \|\hat{Y}^\varepsilon\|_{M_\lambda[t,T]}^2 + (KC_2 + k_1^2) \|\hat{Z}^\varepsilon\|_{M_\lambda[t,T]}^2, \end{aligned} \quad (28)$$

$$\|\hat{Y}^\varepsilon\|_{M_\lambda[t,T]}^2 \leq B(\bar{\lambda}_2, T) \left[k_2^2 e^{-\lambda T} E|\hat{X}_T^\varepsilon|^2 + KC_3 \|\hat{X}^\varepsilon\|_{M_\lambda[t,T]}^2 \right], \quad (29)$$

and

$$\|\hat{Z}^\varepsilon\|_{M_\lambda[t,T]}^2 \leq \frac{A(\bar{\lambda}_2, T)}{\alpha} \left[k_2^2 e^{-\lambda T} E|\hat{X}_T^\varepsilon|^2 + KC_3 \|\hat{X}^\varepsilon\|_{M_\lambda[t,T]}^2 \right]. \quad (30)$$

From these estimates, recalling the definition of $\mu(\alpha, T)$, we get

$$(1 - \mu(\alpha, T)k_2^2) e^{-\lambda T} E|\hat{X}_T^\varepsilon|^2 + (\bar{\lambda}_1 - \mu(\alpha, T)KC_3) \|\hat{X}^\varepsilon\|_{M_\lambda[t,T]}^2 \leq e^{-(\lambda \wedge 0)T} |x_1 - x_2|^2, \quad (31)$$

from where we obtain

$$e^{-\lambda T} E|\hat{X}_T^\varepsilon|^2 + \|\hat{X}^\varepsilon\|_{M_\lambda[t,T]}^2 \leq C_T |x_1 - x_2|^2, \quad (32)$$

with

$$C_T = \max \left\{ \frac{e^{-(\lambda \wedge 0)T}}{1 - \mu(\alpha, T)k_2^2}, \frac{e^{-(\lambda \wedge 0)T}}{\bar{\lambda}_1 - \mu(\alpha, T)KC_3} \right\}. \quad (33)$$

Observe that C_T does not depend on (t, x_1, x_2) nor on ε . From (32), (29) and (30), we have

$$\|\hat{X}^\varepsilon\|_{M_\lambda[t,T]} + \|\hat{Y}^\varepsilon\|_{M_\lambda[t,T]} + \|\hat{Z}^\varepsilon\|_{M_\lambda[t,T]} \leq C_T |x_1 - x_2|^2.$$

Here C_T differs from (33), independent of (t, x_1, x_2) and ε and it may vary line by line in the following discussion. Finally, from Itô's formula and the BDG inequality, we conclude that

$$\|\hat{X}^\varepsilon\|_{S[t,T]} + \|\hat{Y}^\varepsilon\|_{S[t,T]} + \|\hat{Z}^\varepsilon\|_{M[t,T]} \leq C_T |x_1 - x_2|^2.$$

The proof is completed now. ■

Remark 18 Similar to Proposition 17 we show that for all $\varepsilon > 0$, $0 \leq t \leq T$ and for all $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, there is some constant C_T independent of (t, ξ_1, ξ_2) and ε , such that

$$\begin{aligned} & E^{\mathcal{F}_t} \left(\sup_{t \leq r \leq T} |X_r^{\varepsilon, t, \xi_1} - X_r^{\varepsilon, t, \xi_2}|^2 + \sup_{t \leq r \leq T} |Y_r^{\varepsilon, t, \xi_1} - Y_r^{\varepsilon, t, \xi_2}|^2 + \int_t^T |Z_r^{\varepsilon, t, \xi_1} - Z_r^{\varepsilon, t, \xi_2}|^2 dr \right) \\ & \leq C_T |\xi_1 - \xi_2|^2, \quad \mathbb{P} - a.s. \end{aligned}$$

In particular, $|Y_t^{\varepsilon, t, \xi_1} - Y_t^{\varepsilon, t, \xi_2}| \leq C_T |\xi_1 - \xi_2|$, a.s.

Now we introduce the random field $\theta^\varepsilon(t, x) := Y_t^{\varepsilon, t, x}$, for $(t, x) \in [0, T] \times \mathbb{R}^n$. Then

$$|\theta^\varepsilon(t, x) - \theta^\varepsilon(t, x')| \leq C_T |x - x'|, \quad a.s. \quad (34)$$

Moreover, we have the following proposition:

Proposition 19 Let us suppose the assumptions (H_1) – (H_5) as well as $(C1)$ combined either with $(C2)$ or with $(C3)$ for some $\lambda, \alpha, C_1, C_2, C_3$ and $C_4 = \frac{1-\alpha}{K}$. Then, for any $t \in [0, T]$, and $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^m)$, we have

$$\theta^\varepsilon(t, \zeta) = Y_t^{\varepsilon, t, \zeta}, \quad \mathbb{P} - a.s.$$

The proof of the above Proposition can be obtained by combining the arguments of Peng ([23], Theorem 4.7) with the uniqueness of the solution of our penalized FBSDE.

In the following discussion, we recall the assumption:

(H'_3) $k_1 = 0$, i.e., σ does not depend on z .

Under (H'_3) , FBSVI (11) becomes

$$\begin{cases} dX_r + \partial\psi(X_r)dr \ni b(r, X_r, Y_r, Z_r)dr + \sigma(r, X_r, Y_r)dB_r, \\ -dY_r + \partial\varphi(Y_r)dr \ni f(r, X_r, Y_r, Z_r)dr - Z_r dB_r, \quad r \in [0, T], \\ X_0 = x, \quad Y_T = g(X_T), \end{cases} \quad (35)$$

and the corresponding penalized FBSDE is

$$\begin{cases} X_s^\varepsilon + \int_0^s \nabla\psi_\varepsilon(X_r^\varepsilon)dr = x + \int_0^s b(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon)dr + \int_0^s \sigma(r, X_r^\varepsilon, Y_r^\varepsilon)dB_r, \\ Y_s^\varepsilon + \int_s^T \nabla\varphi_\varepsilon(Y_r^\varepsilon)dr = g(X_T^\varepsilon) + \int_s^T f(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon)dr - \int_s^T Z_r^\varepsilon dB_r. \end{cases} \quad (36)$$

Unlike [9], we need the following uniform L^p -estimates of the solution of (36) in our framework:

Proposition 20 Let the assumptions (H_1) – (H_5) and $(H'_4), (H'_5)$ be satisfied. We also assume $(C1)$ and either $(C2)$ or $(C3)$ hold for some $\lambda, \alpha, C_1, C_2, C_3$ and $C_4 = \frac{1-\alpha}{K}$. Then, for every $1 \leq p \leq \frac{3+\rho_0}{2}$, there exists a constant C independent of ε and x , such that

$$\begin{aligned} & E \left(\sup_{0 \leq r \leq T} |X_r^\varepsilon|^{2p} + \sup_{0 \leq r \leq T} |Y_r^\varepsilon|^{2p} \right) + E \left\{ \left(\int_0^T |Z_r^\varepsilon|^2 dr \right)^p \right\} \\ & \leq CE \left\{ |x|^{2p} + |g^0|^{2p} + \left(\int_0^T |b^0(r)|^2 dr \right)^p + \left(\int_0^T |f^0(r)|^2 dr \right)^p + \left(\int_0^T |\sigma^0(r)|^2 dr \right)^p \right\}. \end{aligned} \quad (37)$$

Proof. For the proof, we use an approach based on Theorem A.5 Delarue [10]. But unlike [10], our coefficients ψ_ε and φ_ε depend on ε so that we have to pay some special care. Let us give a sketch of the proof.

Given $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, we construct the following sequence $\{(X^{\varepsilon,k}, Y^{\varepsilon,k}, Z^{\varepsilon,k})\}_{k \geq 1}$ of processes:

$$\left\{ \begin{array}{l} X_s^{\varepsilon,k+1} + \int_t^s \nabla \psi_\varepsilon(X_r^{\varepsilon,k+1}) dr = \xi \\ \quad + \int_t^s b(r, X_r^{\varepsilon,k+1}, Y_r^{\varepsilon,k}, Z_r^{\varepsilon,k}) dr + \int_t^s \sigma(r, X_r^{\varepsilon,k+1}, Y_r^{\varepsilon,k}) dB_r, \\ Y_s^{\varepsilon,k+1} + \int_s^T \nabla \varphi_\varepsilon(Y_r^{\varepsilon,k+1}) dr = g(X_T^{\varepsilon,k+1}) \\ \quad + \int_t^T f(r, X_r^{\varepsilon,k+1}, Y_r^{\varepsilon,k+1}, Z_r^{\varepsilon,k+1}) dr - \int_t^T Z_r^{\varepsilon,k+1} dB_r, \quad s \in [t, T]. \end{array} \right. \quad (38)$$

If we choose $(X^{\varepsilon,0}, Y^{\varepsilon,0}, Z^{\varepsilon,0})$ s.t. $E(\sup_{t \leq r \leq T} |X_r^{\varepsilon,0}|^{2p} + \sup_{t \leq r \leq T} |Y_r^{\varepsilon,0}|^{2p}) + E\left(\int_t^T |Z_r^{\varepsilon,0}|^2 dr\right)^p < \infty$, following the argument at page 264-265 [10], by using Proposition 31 (see the appendix), we obtain the existence of a constant $\delta_{K,k_2,\gamma,p}$ small enough, such that for all $T-t \leq \delta_{K,k_2,\gamma,p} \wedge T_0$ (T_0 is the constant depending on K, γ, k_1, k_2 chosen as in Theorem 30 of the appendix), we have

$$E \sup_{t \leq r \leq T} |X_r^{\varepsilon,k} - X_r^{\varepsilon,l}|^{2p} + E \sup_{t \leq r \leq T} |Y_r^{\varepsilon,k} - Y_r^{\varepsilon,l}|^{2p} + E\left(\int_t^T |Z_r^{\varepsilon,k} - Z_r^{\varepsilon,l}|^2 dr\right)^p \rightarrow 0,$$

as $k, l \rightarrow \infty$. This means that $(X^{\varepsilon,k}, Y^{\varepsilon,k}, Z^{\varepsilon,k})$ converges to some $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ in the sense that

$$E \sup_{t \leq r \leq T} |X_r^{\varepsilon,k} - X_r^\varepsilon|^{2p} + E \sup_{t \leq r \leq T} |Y_r^{\varepsilon,k} - Y_r^\varepsilon|^{2p} + E\left(\int_t^T |Z_r^{\varepsilon,k} - Z_r^\varepsilon|^2 dr\right)^p \rightarrow 0. \quad (39)$$

Then (38), (39) and Theorem 30 yield that $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ is the unique solution of FBSDE:

$$\left\{ \begin{array}{l} X_s^\varepsilon + \int_t^s \nabla \psi_\varepsilon(X_r^\varepsilon) dr = \xi + \int_t^s b(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) dr + \int_t^s \sigma(r, X_r^\varepsilon, Y_r^\varepsilon) dB_r, \\ Y_s^\varepsilon + \int_s^T \nabla \varphi_\varepsilon(Y_r^\varepsilon) dr = g(X_T^\varepsilon) + \int_t^T f(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) dr - \int_t^T Z_r^\varepsilon dB_r, \quad s \in [t, T]. \end{array} \right. \quad (40)$$

Moreover from (39), it follows that, for $T-t \leq \delta_{K,k_2,\gamma,p} \wedge T_0$,

$$E(\sup_{t \leq r \leq T} |X_r^\varepsilon|^{2p} + \sup_{t \leq r \leq T} |Y_r^\varepsilon|^{2p}) + E\left(\int_t^T |Z_r^\varepsilon|^2 dr\right)^p < \infty.$$

Now applying (85) for $(X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon)$ and $(0, 0, 0)$ (Obviously that $(0, 0, 0)$ is a solution of FBSDE (40) with parameters $(\xi, b, \sigma, f, g) = (0, 0, 0, 0, 0)$), we see there exists $0 < \delta'_{K,k_2,\gamma,p} \leq$

$\delta_{K,k_2,\gamma,p}$ small enough and a constant $C_{K,k_2,\gamma,p}$ such that, for $T - t \leq \delta'_{K,k_2,\gamma,p} \wedge T_0$,

$$\begin{aligned} & E\left(\sup_{t \leq r \leq T} |X_r^\varepsilon|^{2p} + \sup_{t \leq r \leq T} |Y_r^\varepsilon|^{2p}\right) + E\left(\int_t^T |Z_r^\varepsilon|^2 dr\right)^p \\ & \leq C_{K,k_2,\gamma,p} E\left\{|\xi|^{2p} + |g^0|^{2p} + \left(\int_t^T |\sigma^0(r)|^2 dr\right)^p + \left(\int_t^T (|b^0(r)| + |f^0(r)|) dr\right)^{2p}\right\}. \end{aligned} \quad (41)$$

If we take $t = 0$, we know that the above inequality yields our result for T small enough ($T \leq \delta'_{K,k_2,\gamma,p} \wedge T_0$). In what follows, we will show that this inequality can be extended to the whole interval.

We use the notation $\theta^\varepsilon(t, x) = Y_t^{\varepsilon,t,x}$, $(t, x) \in [0, T] \times \mathbb{R}^n$. From (34), it follows that for all $t \in [0, T]$,

$$|\theta^\varepsilon(t, x) - \theta^\varepsilon(t, x')| \leq C_T |x - x'|, \text{ a.s.}$$

Now we divide the interval $[0, T]$ into subintervals defined by $(t_i)_{i=0,\dots,N}$ where $t_i = \frac{iT}{N}$, $N \in \mathbb{N}$, such that $\frac{T}{N} \leq \delta'_{K,C_T,\gamma,p} \wedge T_0$, where T_0 is a constant as in Theorem 30, but corresponding to K, γ, k_1, C_T . From Theorem 30 we know that $Y_{t_{i+1}}^{\varepsilon,0,x} = Y_{t_{i+1}}^{\varepsilon,t_{i+1},X_{t_{i+1}}^{\varepsilon,0,x}}$. Then Proposition 19 yields

$$\theta^\varepsilon(t_{i+1}, X_{t_{i+1}}^{\varepsilon,0,x}) = Y_{t_{i+1}}^{\varepsilon,t_{i+1},X_{t_{i+1}}^{\varepsilon,0,x}} = Y_{t_{i+1}}^{\varepsilon,0,x}, \text{ } \mathbb{P} - a.s.$$

The above discussion and (41) allow to follow the argument developed at page 266 [10] to obtain by induction that

$$\begin{aligned} & E\left(\sup_{0 \leq r \leq T} |X_r^\varepsilon|^{2p} + \sup_{0 \leq r \leq T} |Y_r^\varepsilon|^{2p}\right) + E\left(\int_0^T |Z_r^\varepsilon|^2 dr\right)^p \\ & \leq C_{K,C_T,\gamma,p} E\left\{|x|^{2p} + |g^0|^{2p} + \left(\int_0^T |\sigma^0(r)|^2 dr\right)^p + \left(\int_0^T (|b^0(r)| + |f^0(r)|) dr\right)^{2p}\right\}. \end{aligned}$$

■

4.4 Existence and uniqueness of solutions of FBSVIs

In this subsection, we study the existence and the uniqueness of the solution of FBSVI (11). For this, we shall introduce the following condition:

(H₆) There exists a random variable $\zeta \in L^1(\Omega)$ and a constant L such that

$$|\varphi(g(\omega, x))| \leq \zeta(\omega) + L|x|^{3+\rho_0}, \text{ a.s.}$$

Based on the idea of [19] and [20], we give the following auxiliary proposition:

Proposition 21 *Let the assumptions (H₁) – (H₆) and (H'₄), (H'₅) be satisfied. Assume also (C1) and either (C2) or (C3) hold for some $\lambda, \alpha, C_1, C_2, C_3$ and $C_4 = \frac{1-\alpha}{K}$. Then for all*

$$0 < \rho \leq \frac{1+\rho_0}{4} \wedge 1,$$

$$(i) \quad E \sup_{0 \leq r \leq T} |\nabla \psi_\varepsilon(X_r^\varepsilon)|^{2+4\rho} \leq \frac{C}{\varepsilon^{2+3\rho}},$$

$$(ii) \quad E \sup_{0 \leq r \leq T} |X_r^\varepsilon - J_{\varepsilon, \psi}(X_r^\varepsilon)|^{2+4\rho} \leq C\varepsilon^\rho,$$

$$(iii) \quad E \int_s^T |\nabla \varphi_\varepsilon(Y_r^\varepsilon)|^2 dr \leq C, \quad \text{for all } s \in [0, T],$$

$$(iv) \quad E\varphi(J_{\varepsilon, \varphi}(Y_s^\varepsilon)) + E \int_s^T \varphi(J_{\varepsilon, \varphi}(Y_r^\varepsilon)) dr \leq C, \quad \text{for all } s \in [0, T],$$

$$(v) \quad E|Y_s^\varepsilon - J_{\varepsilon, \varphi}(Y_s^\varepsilon)|^2 \leq \varepsilon C, \quad \text{for all } s \in [0, T],$$

Proof. We first prove (iii) – (v). Similar to Proposition 2.2 [19], we can obtain

$$\begin{aligned} e^{-\lambda s} \varphi_\varepsilon(Y_s^\varepsilon) + \int_s^T e^{-\lambda r} |\nabla \varphi_\varepsilon(Y_r^\varepsilon)|^2 dr &\leq e^{-\lambda T} \varphi_\varepsilon(Y_T^\varepsilon) - \int_s^T e^{-\lambda r} \langle \nabla \varphi_\varepsilon(Y_r^\varepsilon), Z_r^\varepsilon dB_r \rangle \\ &+ \int_s^T e^{-\lambda r} \langle \nabla \varphi_\varepsilon(Y_r^\varepsilon), |\lambda| Y_r^\varepsilon + f(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) \rangle dr, \end{aligned} \quad (42)$$

and then

$$\begin{aligned} \frac{1}{2} \int_s^T e^{-\lambda r} E |\nabla \varphi_\varepsilon(Y_r^\varepsilon)|^2 dr &\leq e^{-\lambda T} E \varphi(g(X_T^\varepsilon)) \\ &+ E \int_s^T e^{-\lambda r} [4|\eta(t)|^2 + 4K^2(|X_r^\varepsilon|^2 + |Z_r^\varepsilon|^2) + (4L^2 + |\lambda|^2)|Y_r^\varepsilon|^2] dr. \end{aligned}$$

By using Proposition 20 and (H_6) , we have $\int_s^T e^{-\lambda r} E |\nabla \varphi_\varepsilon(Y_r^\varepsilon)|^2 dr \leq C(1 + |x|^{3+\rho_0})$, which yields (iii). Here C is a constant independent of ε and x .

From (42), (iii) and $\varphi(J_{\varepsilon, \varphi}(y)) \leq \varphi_\varepsilon(y)$, we get $E e^{-\lambda t} \varphi(J_{\varepsilon, \varphi}(Y_t^\varepsilon)) \leq C$, for any $t \in [0, T]$. Thus (iv) follows easily.

Moreover, since $|y - J_{\varepsilon, \varphi}(y)|^2 = |\nabla \varphi_\varepsilon(y)|^2 \leq 2\varepsilon \varphi_\varepsilon(y)$, $y \in \mathbb{R}^m$, (v) is obtained from (iv).

Now we are focusing on the proof of (i) and (ii). We shall follow the argument as in Theorem 4.20 [20], so we only give a sketch of the proof here. Since ψ_ε is a function of class $C^1(\mathbb{R}^n; \mathbb{R}_+)$ and the gradient $\nabla \psi_\varepsilon(u)$ is a Lipschitz function with Lipschitz constant $1/\varepsilon$, from Remark 33 (see the appendix), we have

$$\begin{aligned} &\psi_\varepsilon^{1+2\rho}(X_s^\varepsilon) + (1+2\rho) \int_0^s \psi_\varepsilon^{2\rho}(X_r^\varepsilon) |\nabla \psi_\varepsilon(X_r^\varepsilon)|^2 dr \\ &\leq \psi_\varepsilon^{1+2\rho}(x) + (1+2\rho) \int_0^s \psi_\varepsilon^{2\rho}(X_r^\varepsilon) \langle \nabla \psi_\varepsilon(X_r^\varepsilon), b(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) \rangle dr \\ &\quad + \frac{1+2\rho}{2\varepsilon} \int_0^s \psi_\varepsilon^{2\rho}(X_r^\varepsilon) |\sigma(r, X_r^\varepsilon, Y_r^\varepsilon)|^2 dr \\ &\quad + \rho(1+2\rho) \int_0^s \psi_\varepsilon^{2\rho-1}(X_r^\varepsilon) |\nabla \psi_\varepsilon(X_r^\varepsilon)|^2 |\sigma(r, X_r^\varepsilon, Y_r^\varepsilon)|^2 dr \\ &\quad + (1+2\rho) \int_0^s \psi_\varepsilon^{2\rho}(X_r^\varepsilon) \langle \nabla \psi_\varepsilon(X_r^\varepsilon), \sigma(r, X_r^\varepsilon, Y_r^\varepsilon) dB_r \rangle. \end{aligned}$$

Then, similarly, following the argument as in Theorem 4.20 [20], we can prove that there exists a constant C independent of ε and x , such that

$$E \sup_{0 \leq r \leq T} \psi_\varepsilon^{1+2\rho}(X_r^\varepsilon) + E \int_0^T \psi_\varepsilon^{2\rho}(X_r^\varepsilon) |\nabla \psi_\varepsilon(X_r^\varepsilon)|^2 dr \leq 2\psi_\varepsilon^{1+2\rho}(x) + \frac{C}{\varepsilon} E \int_0^T \psi_\varepsilon^{2\rho}(X_r^\varepsilon) A_r^\varepsilon dr. \quad (43)$$

Here

$$A_r^\varepsilon = |b^0(r)|^2 + |\sigma^0(r)|^2 + |X_r^\varepsilon|^2 + |Y_r^\varepsilon|^2 + |X_r^\varepsilon| |Z_r^\varepsilon| \quad r \in [0, T].$$

Moreover, let $r_0 = \frac{1+\rho}{\rho}$. Then from Young's inequality,

$$\begin{aligned} \frac{C}{\varepsilon} \psi_\varepsilon^{2\rho}(X_r^\varepsilon) A_r^\varepsilon &= \psi_\varepsilon^{2\rho - \frac{2}{r_0}}(X_r^\varepsilon) \psi_\varepsilon^{\frac{2}{r_0}}(X_r^\varepsilon) A_r^\varepsilon \frac{C}{\varepsilon} \leq \psi_\varepsilon^{2\rho - \frac{2}{r_0}}(X_r^\varepsilon) |\nabla \psi_\varepsilon(X_r^\varepsilon)|^{\frac{2}{r_0}} |X_r^\varepsilon|^{\frac{2}{r_0}} A_r^\varepsilon \frac{C}{\varepsilon} \\ &\leq \frac{1}{r_0} \left(\psi_\varepsilon^{2\rho - \frac{2}{r_0}}(X_r^\varepsilon) |\nabla \psi_\varepsilon(X_r^\varepsilon)|^{\frac{2}{r_0}} \right)^{r_0} + \frac{r_0-1}{r_0} \left(|X_r^\varepsilon|^{\frac{2}{r_0}} A_r^\varepsilon \frac{C}{\varepsilon} \right)^{\frac{r_0}{r_0-1}} \\ &= \frac{\rho}{1+\rho} \psi_\varepsilon^{2\rho}(X_r^\varepsilon) |\nabla \psi_\varepsilon(X_r^\varepsilon)|^2 + \frac{1}{\varepsilon^{1+\rho}} \frac{C^{1+\rho}}{1+\rho} |X_r^\varepsilon|^{2\rho} |A_r^\varepsilon|^{1+\rho}. \end{aligned}$$

From the above estimate, (37), (43) and Young's inequality it follows that

$$\begin{aligned} E \sup_{0 \leq r \leq T} \psi_\varepsilon^{1+2\rho}(X_r^\varepsilon) &\leq 2\psi_\varepsilon^{1+2\rho}(x) + \frac{C}{\varepsilon^{1+\rho}} E \int_0^T |X_r^\varepsilon|^{2\rho} |A_r^\varepsilon|^{1+\rho} dr \leq 2\psi_\varepsilon^{1+2\rho}(x) \\ &\quad + \frac{C}{\varepsilon^{1+\rho}} E \int_0^T \left(|b^0(r)|^{2+4\rho} + |\sigma^0(r)|^{2+4\rho} + |X_r^\varepsilon|^{2+4\rho} + |Y_r^\varepsilon|^{2+4\rho} + |X_r^\varepsilon|^{1+3\rho} |Z_r^\varepsilon|^{1+\rho} \right) dr \\ &\leq 2\psi_\varepsilon^{1+2\rho}(x) + \frac{C}{\varepsilon^{1+\rho}} \left[E \sup_{0 \leq r \leq T} |X_r^\varepsilon|^{2+4\rho} + E \sup_{0 \leq r \leq T} |Y_r^\varepsilon|^{2+4\rho} \right. \\ &\quad \left. + E \int_0^T \left(|b^0(r)|^{2+4\rho} + |\sigma^0(r)|^{2+4\rho} \right) dr + E \left\{ \sup_{0 \leq r \leq T} |X_r^\varepsilon|^{1+3\rho} \left(\int_0^T |Z_r^\varepsilon|^2 dr \right)^{\frac{1+\rho}{2}} \right\} \right] \\ &\leq 2\psi_\varepsilon^{1+2\rho}(x) + \frac{C}{\varepsilon^{1+\rho}} \left[E \sup_{0 \leq r \leq T} |X_r^\varepsilon|^{2+4\rho} + E \sup_{0 \leq r \leq T} |Y_r^\varepsilon|^{2+4\rho} \right. \\ &\quad \left. + E \int_0^T \left(|b^0(r)|^{2+4\rho} + |\sigma^0(r)|^{2+4\rho} \right) dr + E \left\{ \left(\int_0^T |Z_r^\varepsilon|^2 dr \right)^{1+2\rho} \right\} \right] \\ &\leq \frac{C}{\varepsilon^{1+\rho}} \left(1 + |x|^{3+\rho_0} + \psi^{1+2\rho}(x) \right), \end{aligned} \quad (44)$$

where C is a constant independent of ε and x and it can vary from line to line. (We can assume that $\varepsilon < 1$. Also recall that $0 \leq \rho \leq \frac{1+\rho_0}{4} \wedge 1$ so that $2+4\rho \leq 3+\rho_0$).

Finally, since $\frac{\varepsilon}{2} |\nabla \psi_\varepsilon(X_r^\varepsilon)|^2 \leq \psi_\varepsilon(X_r^\varepsilon)$ and $X_r^\varepsilon - J_{\varepsilon, \psi}(X_r^\varepsilon) = \varepsilon \nabla \psi_\varepsilon(X_r^\varepsilon)$, we obtain (i) and (ii) directly from (44). \blacksquare

Proposition 22 *Under the assumptions of Proposition 21, setting $\frac{1-\rho_0}{4+4\rho_0} \vee 0 < \rho \leq \frac{1+\rho_0}{4} \wedge 1$,*

we have

$$\begin{aligned}
(i) \quad & E \int_0^T \left\{ |X_r^{\varepsilon_1} - X_r^{\varepsilon_2}|^2 + |Y_r^{\varepsilon_1} - Y_r^{\varepsilon_2}|^2 + |Z_r^{\varepsilon_1} - Z_r^{\varepsilon_2}|^2 \right\} dr \leq C \left(\varepsilon_1^{\frac{\rho}{2+4\rho}} + \varepsilon_2^{\frac{\rho}{2+4\rho}} \right), \\
(ii) \quad & E \left\{ \sup_{0 \leq r \leq T} |X_r^{\varepsilon_1} - X_r^{\varepsilon_2}|^2 + \sup_{0 \leq r \leq T} |Y_r^{\varepsilon_1} - Y_r^{\varepsilon_2}|^2 \right\} \leq C \left(\varepsilon_1^{\frac{\rho}{2+4\rho}} + \varepsilon_2^{\frac{\rho}{2+4\rho}} \right).
\end{aligned}$$

Here C is a constant which depends neither on ε_1 nor on ε_2 .

Proof. From Proposition 16 we have (23). In the same manner as in [20], applying the Hölder inequality to the right-hand side of (23), it follows that

$$\begin{aligned}
& (\varepsilon_1 + \varepsilon_2) E \int_0^T |\nabla \psi_{\varepsilon_1}(X_r^{\varepsilon_1})| |\nabla \psi_{\varepsilon_2}(X_r^{\varepsilon_2})| dr \\
& \leq \varepsilon_1 \left(E \sup_{0 \leq r \leq T} |\nabla \psi_{\varepsilon_1}(X_r^{\varepsilon_1})|^{2+4\rho} \right)^{\frac{1}{2+4\rho}} \left[E \left(\int_0^T |\nabla \psi_{\varepsilon_2}(X_r^{\varepsilon_2})| dr \right)^{\frac{2+4\rho}{1+4\rho}} \right]^{\frac{1+4\rho}{2+4\rho}} \\
& \quad + \varepsilon_2 \left(E \sup_{0 \leq r \leq T} |\nabla \psi_{\varepsilon_2}(X_r^{\varepsilon_2})|^{2+4\rho} \right)^{\frac{1}{2+4\rho}} \left[E \left(\int_0^T |\nabla \psi_{\varepsilon_1}(X_r^{\varepsilon_1})| dr \right)^{\frac{2+4\rho}{1+4\rho}} \right]^{\frac{1+4\rho}{2+4\rho}}.
\end{aligned} \tag{45}$$

Now we calculate $E \left\{ \left(\int_0^T |\nabla \psi_{\varepsilon}(X_r^{\varepsilon})| dr \right)^{\frac{2+4\rho}{1+4\rho}} \right\}$. From (14) we know

$$2r_0 \int_0^T |\nabla \psi_{\varepsilon}(X_s^{\varepsilon})| ds \leq 2 \int_0^T \langle \nabla \psi_{\varepsilon}(X_s^{\varepsilon}), X_s^{\varepsilon} - u_0 \rangle ds + 2M_0 T. \tag{46}$$

By applying Itô's formula to $|X_r^{\varepsilon} - u_0|^2$, we obtain

$$\begin{aligned}
& |X_s^{\varepsilon} - u_0|^2 + 2 \int_0^s \langle \nabla \psi_{\varepsilon}(X_r^{\varepsilon}), X_r^{\varepsilon} - u_0 \rangle dr = |x - u_0|^2 + \int_0^s |\sigma(r, X_r^{\varepsilon}, Y_r^{\varepsilon})|^2 dr \\
& \quad + \int_0^s 2 \langle X_r^{\varepsilon} - u_0, b(r, X_r^{\varepsilon}, Y_r^{\varepsilon}, Z_r^{\varepsilon}) \rangle dr + \int_0^s 2 \langle X_r^{\varepsilon} - u_0, \sigma(r, X_r^{\varepsilon}, Y_r^{\varepsilon}) dB_r \rangle.
\end{aligned} \tag{47}$$

Thus

$$\begin{aligned}
2r_0 \int_0^T |\nabla \psi_{\varepsilon}(X_r^{\varepsilon})| dr & \leq 2|x|^2 + 2|u_0|^2 + 2M_0 T + \int_0^T |\sigma(r, X_r^{\varepsilon}, Y_r^{\varepsilon})|^2 dr \\
& \quad + \int_0^T 2 \langle X_r^{\varepsilon} - u_0, b(r, X_r^{\varepsilon}, Y_r^{\varepsilon}, Z_r^{\varepsilon}) \rangle dr + \int_0^T 2 \langle X_r^{\varepsilon} - u_0, \sigma(r, X_r^{\varepsilon}, Y_r^{\varepsilon}) dB_r \rangle.
\end{aligned}$$

Consequently, for fixed $u_0 \in \mathbb{R}^n$, we deduce from Proposition 20, that for all $1 \leq q \leq \frac{3+\rho_0}{2}$,

$$\begin{aligned}
& E \left\{ \left(2r_0 \int_0^T |\nabla \psi_{\varepsilon}(X_r^{\varepsilon})| dr \right)^q \right\} \leq C \left[1 + E \sup_{0 \leq r \leq T} |X_r^{\varepsilon}|^{2q} + E \sup_{0 \leq r \leq T} |Y_r^{\varepsilon}|^{2q} \right. \\
& \quad \left. + E \left\{ \left(\int_0^T |Z_r^{\varepsilon}|^2 dr \right)^q \right\} + E \left(\int_0^T |b^0(r)|^2 dr \right)^q + E \left(\int_0^T |\sigma^0(r)|^2 dr \right)^q \right] \leq C.
\end{aligned} \tag{48}$$

We choose now $q = \frac{2+4\rho}{1+4\rho}$ and we observe that $q \leq \frac{3+\rho_0}{2}$. (Indeed, recall that $\rho \geq \frac{1-\rho_0}{4+4\rho_0} \vee 0$). Then we obtain from (45), (48) and Proposition 21 (i), that

$$\begin{aligned} & (\varepsilon_1 + \varepsilon_2) E \int_0^T |\nabla \psi_{\varepsilon_1}(X_r^{\varepsilon_1})| |\nabla \psi_{\varepsilon_2}(X_r^{\varepsilon_2})| dr \\ & \leq C\varepsilon_1 \left(E \sup_{0 \leq r \leq T} |\nabla \psi_{\varepsilon_1}(X_r^{\varepsilon_1})|^{2+4\rho} \right)^{\frac{1}{2+4\rho}} + C\varepsilon_2 \left(E \sup_{0 \leq r \leq T} |\nabla \psi_{\varepsilon_2}(X_r^{\varepsilon_2})|^{2+4\rho} \right)^{\frac{1}{2+4\rho}} \\ & \leq C\varepsilon_1 \left(\frac{C}{\varepsilon_1^{2+3\rho}} \right)^{\frac{1}{2+4\rho}} + C\varepsilon_2 \left(\frac{C}{\varepsilon_2^{2+3\rho}} \right)^{\frac{1}{2+4\rho}} \leq C\varepsilon_1^{\frac{\rho}{2+4\rho}} + C\varepsilon_2^{\frac{\rho}{2+4\rho}} \end{aligned} \quad (49)$$

On the other hand, using Proposition 21 (iii), it follows

$$\begin{aligned} & (\varepsilon_1 + \varepsilon_2) E \int_0^T |\nabla \varphi_{\varepsilon_1}(X_r^{\varepsilon_1})| |\nabla \varphi_{\varepsilon_2}(X_r^{\varepsilon_2})| dr \\ & \leq \frac{\varepsilon_1 + \varepsilon_2}{2} E \int_0^T (|\nabla \varphi_{\varepsilon_1}(X_r^{\varepsilon_1})|^2 + |\nabla \varphi_{\varepsilon_2}(X_r^{\varepsilon_2})|^2) dr \leq C(\varepsilon_1 + \varepsilon_2). \end{aligned} \quad (50)$$

Consequently, (23), (49) and (50) allow to complete the proof. \blacksquare

Now we are able to give our main results:

Theorem 23 Suppose $(H_1) - (H_6)$ and (H'_5) are satisfied and (H'_4) holds with $\rho_0 \geq 1$. Moreover, we assume that (C1) and either (C2) or (C3) hold for some $\lambda, \alpha, C_1, C_2, C_3$ and $C_4 = \frac{1-\alpha}{K}$. Then there exists a unique solution (X, Y, Z, V, U) of FBSVI (35).

Proof. The uniqueness is a consequence of Proposition 14. Thus, it remains to show the existence. From Proposition 22, we know that there exist $X \in S_n^2$, $Y \in S_m^2$, and $Z \in M_{m \times d}^2$, such that

$$\lim_{\varepsilon \rightarrow 0} E \left\{ \sup_{0 \leq r \leq T} |X_r^\varepsilon - X_r|^2 + \sup_{0 \leq r \leq T} |Y_r^\varepsilon - Y_r|^2 + \int_0^T |Z_r^\varepsilon - Z_r|^2 dr \right\} = 0, \quad (51)$$

and from Proposition 21 (ii) and (v), we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} E \sup_{0 \leq r \leq T} |J_{\varepsilon, \psi}(X_r^\varepsilon) - X_r|^2 = 0, \quad \lim_{\varepsilon \rightarrow 0} E |Y_r - J_{\varepsilon, \varphi}(Y_r^\varepsilon)|^2 = 0, \text{ for all } r \in [0, T], \\ & \text{and } \lim_{\varepsilon \rightarrow 0} \int_0^T E |Y_r - J_{\varepsilon, \varphi}(Y_r^\varepsilon)|^2 dr = 0. \end{aligned} \quad (52)$$

Let us define $V_s^\varepsilon := \int_0^s \nabla \psi_\varepsilon(X_r^\varepsilon) dr$. Then from (36),

$$X_s^\varepsilon + V_s^\varepsilon = x + \int_0^s b(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) dr + \int_0^s \sigma(r, X_r^\varepsilon, Y_r^\varepsilon) dB_r, \quad s \in [0, T].$$

The Lipschitz condition for b and σ , Proposition 22 and the BDG inequality yield that

$$E \sup_{0 \leq s \leq T} |V_s^{\varepsilon_1} - V_s^{\varepsilon_2}|^2 \leq C \left(\varepsilon_1^{\frac{\rho}{2+4\rho}} + \varepsilon_2^{\frac{\rho}{2+4\rho}} \right), \quad \varepsilon_1, \varepsilon_2 > 0.$$

Consequently, there exists $V \in S_n^2$, such that

$$\lim_{\varepsilon \rightarrow 0} E \sup_{0 \leq s \leq T} |V_s^\varepsilon - V_s|^2 = 0 \quad (53)$$

and

$$X_s + V_s = x + \int_0^s b(r, X_r, Y_r, Z_r) dr + \int_0^s \sigma(r, X_r, Y_r) dB_r.$$

Then, from (48) and $V^\varepsilon(0) = 0$ we have $E \left\{ \|V^\varepsilon\|_{BV([0,T];\mathbb{R}^n)}^q \right\} = E \uparrow V^\varepsilon \downarrow_{[0,T]}^q \leq C$, for $1 \leq q \leq \frac{3+\rho_0}{2}$. In particular, $E \left\{ \|V^\varepsilon\|_{BV([0,T];\mathbb{R}^n)} \right\} \leq C$. Moreover, if $\rho_0 \geq 1$, then also

$$E \left\{ \|V^\varepsilon\|_{BV([0,T];\mathbb{R}^n)}^2 \right\} \leq C.$$

Recalling Proposition 1.25 [20] (see also Proposition 34 in the appendix) as well as (51) and (53), we have

$$E \left\{ \|V\|_{BV([0,T];\mathbb{R}^n)}^2 \right\} = E \uparrow V \downarrow_{[0,T]}^2 \leq C, \quad (54)$$

and for all $0 \leq s \leq t \leq T$,

$$\int_s^t \langle X_r^\varepsilon, dV_r^\varepsilon \rangle \xrightarrow{\mathbb{P}} \int_s^t \langle X_r, dV_r \rangle, \text{ as } \varepsilon \rightarrow 0. \quad (55)$$

From (13-(a)) and the convexity of ψ_ε , we have

$$\psi(J_{\varepsilon,\psi}(x)) \leq \psi_\varepsilon(x) \leq \psi_\varepsilon(z) + \langle x - z, \nabla \psi_\varepsilon(x) \rangle \leq \psi(z) + \langle x - z, \nabla \psi_\varepsilon(x) \rangle, \quad z \in \mathbb{R}^n,$$

Then for all $0 \leq s \leq t \leq T$, it follows that

$$\int_s^t \psi(J_{\varepsilon,\psi}(X_r^\varepsilon)) dr \leq (t-s)\psi(z) + \int_s^t \langle X_r^\varepsilon - z, dV_r^\varepsilon \rangle, \quad z \in \mathbb{R}^n.$$

From (52), (55), Fatou's lemma and the fact that ψ is l.s.c., letting $\varepsilon \rightarrow 0$, we deduce

$$\int_s^t \psi(X_r) dr \leq (t-s)\psi(z) + \int_s^t \langle X_r - z, dV_r \rangle, \quad z \in \mathbb{R}^n, \quad 0 \leq s \leq t \leq T, \text{ a.s.} \quad (56)$$

This proves the inequality in Definition 9 (a_2). Moreover, taking $z = 0$ in (56), we have

$$0 \leq E \int_0^T \psi(X_r) dr \leq E \left| \int_0^T \langle X_r, dV_r \rangle \right| \leq \frac{1}{2} E \sup_{0 \leq t \leq T} |X_r|^2 + \frac{1}{2} E \uparrow V \downarrow_{[0,T]}^2 < \infty,$$

i.e., X takes its values in $Dom\psi$ and $\psi(X) \in L^1(\Omega \times [0, T]; \mathbb{R})$.

On the other hand, for any $\varepsilon > 0$, we define $U_s^\varepsilon = \nabla \varphi_\varepsilon(Y_s^\varepsilon)$ and $\bar{U}_s^\varepsilon = \int_0^s U_r^\varepsilon dr$. Then from (36) and Proposition 22 we deduce the existence of a $\bar{U} \in S_m^2$, s.t.

$$\lim_{\varepsilon \rightarrow 0} E \sup_{0 \leq s \leq T} |\bar{U}_s^\varepsilon - \bar{U}_s|^2 = 0. \quad (57)$$

Moreover, Proposition 21 (iii) yields $\sup_{\varepsilon>0} \mathbb{E} \int_0^T |U_r^\varepsilon|^2 dr \leq C$. Consequently, the sequence $\{\bar{U}^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\Omega; H^1(0, T))$. Thus there exists a subsequence which converges weakly to a limit in $L^2(\Omega; H^1(0, T))$. But from (57) we conclude that this limit is nothing but \bar{U}_t , and the whole sequence $\{\bar{U}^\varepsilon\}_{\varepsilon>0}$ converges weakly to \bar{U} . Moreover, \bar{U} takes the form $\bar{U}_s = \int_0^s U_r dr$, $s \in [0, T]$, where $U^\varepsilon \xrightarrow[\text{weakly}]{L^2(\Omega \times [0, T])} U$. Now we take the limit in probability in FBSDE (36), and we get

$$Y_s + \int_s^T U_r dr = g(X_T) + \int_s^T f(r, X_r, Y_r, Z_r) dr - \int_s^T Z_r dB_r, \quad \text{for all } s \in [0, T], \text{ a.s..}$$

Finally, it remains to show that $(Y_s, U_s) \in \partial\varphi$. In fact, from $U_s^\varepsilon \in \partial\varphi(J_{\varepsilon, \varphi} Y_s^\varepsilon)$, it follows ,

$$\langle U_s^\varepsilon, v_s - J_{\varepsilon, \varphi}(Y_s^\varepsilon) \rangle + \varphi(J_{\varepsilon, \varphi}(Y_s^\varepsilon)) \leq \varphi(v_s) \quad \text{for all } v \in M_m^2.$$

Then integrating both sides from a to b , for all $0 \leq a \leq b \leq T$, we obtain

$$\int_a^b \langle U_s^\varepsilon, v_s - J_{\varepsilon, \varphi}(Y_s^\varepsilon) \rangle ds + \int_a^b \varphi(J_{\varepsilon, \varphi}(Y_s^\varepsilon)) ds \leq \int_a^b \varphi(v_s) ds.$$

Let us take now the limit as $\varepsilon \rightarrow 0$. By using (52), the weak convergence of U^ε to U as well as the fact that φ is a proper convex l.s.c. function, we get

$$\int_a^b \langle U_s, v_s - Y_s \rangle ds + \int_a^b \varphi(Y_s) ds \leq \int_a^b \varphi(v_s) ds. \quad (58)$$

Indeed, $\int_a^b \langle U_s^\varepsilon, v_s - Y_s^\varepsilon \rangle ds \xrightarrow{\mathbb{P}} \int_a^b \langle U_s, v_s - Y_s \rangle ds$ and $\int_a^b \langle U_s^\varepsilon, J_{\varepsilon, \varphi}(Y_s^\varepsilon) - Y_s^\varepsilon \rangle ds \xrightarrow{\mathbb{P}} 0$. Considering that a, b, v are arbitrary, we conclude from (58) that $(Y_s, U_s) \in \partial\varphi$, $d\mathbb{P} \otimes dt$ a.e. ■

5 Existence of viscosity solutions of PVI

In this section we will prove that the solution of our FBSVI provides a probabilistic interpretation for the solution of PVI (1). For this we assume that

(H₇) The coefficients b, σ, f, g are all deterministic and jointly continuous and $m = 1$.

We collect the following assumptions which we denote by (A1):

(A1) The conditions $(H_1) - (H_7), (H'_5)$ are satisfied and (H'_4) holds with $\rho_0 \geq 1$. Moreover, (C1) and either (C2) or (C3) hold true for some $\lambda, \alpha, C_1, C_2, C_3$ and $C_4 = \frac{1-\alpha}{K}$.

For each $(t, x) \in [0, T] \times \text{Dom}\psi$, we consider the following FBSVI over the interval $[t, T]$:

$$\begin{cases} dX_s + \partial\psi(X_s)ds \ni b(s, X_s, Y_s, Z_s)ds + \sigma(s, X_s, Y_s)dB_s, \\ -dY_s + \partial\varphi(Y_s)ds \ni f(s, X_s, Y_s, Z_s)ds - Z_s dB_s, \quad s \in [t, T], \\ X_t = x, \quad Y_T = g(X_T), \end{cases} \quad (59)$$

It is clearly that Theorem 23 remains true on the interval $[t, T]$, and we denote the unique solution of equation (59) by $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, U_s^{t,x})$. We define

$$u(t, x) := Y_t^{t,x}, \text{ for } t \in [0, T], x \in \text{Dom}\psi. \quad (60)$$

Observe that under (H_7) , $Y_t^{t,x}$ is deterministic. Indeed, by using the standard "time shifting" technique, we can check that $Y_s^{t,x}$ is \mathcal{F}_s^t adapted, where $\mathcal{F}_s^t := \sigma\{B_r : t \leq r \leq s\}$ augmented by the \mathbb{P} -null sets. The function $u(t, x) = Y_t^{t,x}$ has the following properties:

Proposition 24 *Suppose (A1) holds, then $u(t, x) \in \text{Dom}\varphi$, for all $(t, x) \in [0, T] \times \text{Dom}\psi$, and $u \in C([0, T] \times \text{Dom}\psi)$.*

Proof. From Proposition 21 (iv) and the convergence in (52), we deduce that $\varphi(u(t, x)) = E\varphi(Y_t^{t,x}) < \infty$ for all $x \in \text{Dom}\psi$. Consequently, $u(t, x) \in \text{Dom}\varphi$, $(t, x) \in [0, T] \times \text{Dom}\psi$. Let us prove that $u \in C([0, T] \times \text{Dom}\psi)$. We split this proof into two steps.

Step 1: We first prove that u is right continuous w.r.t. t and continuous w.r.t. x . For this we assume that $(t_n, x_n) \rightarrow (t, x)$ as $n \rightarrow \infty$, for $t_n \geq t$. We put $\hat{X}_s^n = X_s^{t_n, x_n} - X_s^{t, x}$, $\hat{Y}_s^n = Y_s^{t_n, x_n} - Y_s^{t, x}$, $\hat{Z}_s^n = Z_s^{t_n, x_n} - Z_s^{t, x}$, $s \in [t_n, T]$. From (19)-(21) of Proposition 14 (Now for $[t_n, T]$), it follows

$$\begin{aligned} e^{-\lambda T} E|\hat{X}_T^n|^2 + \bar{\lambda}_1 \|\hat{X}^n\|_{M_\lambda[t_n, T]}^2 &\leq K(C_1 + K)\|\hat{Y}^n\|_{M_\lambda[t_n, T]}^2 \\ &+ (KC_2 + k_1^2)\|\hat{Z}^n\|_{M_\lambda[t_n, T]}^2 + e^{-\lambda t_n} E|\hat{X}_{t_n}^n|^2, \end{aligned} \quad (61)$$

$$\|\hat{Y}^n\|_{M_\lambda[t_n, T]}^2 \leq B(\bar{\lambda}_2, T) \left[k_2^2 e^{-\lambda T} E|\hat{X}_T^n|^2 + KC_3 \|\hat{X}^n\|_{M_\lambda[t_n, T]}^2 \right], \quad (62)$$

$$\|\hat{Z}^n\|_{M_\lambda[t_n, T]}^2 \leq \frac{A(\bar{\lambda}_2, T)}{\alpha} \left[k_2^2 e^{-\lambda T} E|\hat{X}_T^n|^2 + KC_3 \|\hat{X}^n\|_{M_\lambda[t_n, T]}^2 \right]. \quad (63)$$

Indeed, we observe that here, unlike in (19)-(21), the coefficients $b, \tilde{b}, \sigma, \tilde{\sigma}$ and f, \tilde{f} as well as g, \tilde{g} coincide, so we can consider $\delta \rightarrow 0$ in (19)-(21). In the following C denotes a constant independent of (t, x) and (t_n, x_n) , which may vary from line to line.

Using the compatibility conditions (C1), (C2) or (C1), (C3), we can check with the help of (61)-(63), that

$$|\hat{X}_{\lambda, \beta, [t_n, T]}^n|^2 \leq C e^{-\lambda t_n} E|\hat{X}_{t_n}^n|^2 \leq C(1 + e^{-\lambda T}) E|x_n - X_{t_n}^{t, x}|^2. \quad (64)$$

Then plugging this inequality into (62) and (63), we obtain

$$\|\hat{X}^n\|_{M_\lambda[t_n, T]}^2 + \|\hat{Y}^n\|_{M_\lambda[t_n, T]}^2 + \|\hat{Z}^n\|_{M_\lambda[t_n, T]}^2 \leq CE|x_n - X_{t_n}^{t, x}|^2.$$

By applying BDG inequality to the equations for \hat{X}^n, \hat{Y}^n , we can prove

$$\|\hat{X}^n\|_{S_\lambda[t_n, T]}^2 + \|\hat{Y}^n\|_{S_\lambda[t_n, T]}^2 + \|\hat{Z}^n\|_{M_\lambda[t_n, T]}^2 \leq CE|x_n - X_{t_n}^{t, x}|^2. \quad (65)$$

On the other hand, using Proposition 29, we have

$$\begin{aligned} &\|X_s^{t, x}\|_{S_\lambda[t, T]}^2 + \|Y_s^{t, x}\|_{S_\lambda[t, T]}^2 + \|Z_s^{t, x}\|_{M_\lambda[t, T]}^2 \\ &\leq C \left(e^{-\lambda t} |x|^2 + \|b^0\|_{M_\lambda[t, T]}^2 + \|\sigma^0\|_{M_\lambda[t, T]}^2 + \|f^0\|_{M_\lambda[t, T]}^2 + E|g^0|^2 \right) \\ &\leq C \left(e^{-(\lambda \wedge 0)T} |x|^2 + \|b^0\|_{M_\lambda}^2 + \|\sigma^0\|_{M_\lambda}^2 + \|f^0\|_{M_\lambda}^2 + E|g^0|^2 \right) \\ &\leq C(1 + |x|^2). \end{aligned}$$

Then we deduce from

$$X_s^{t,x} - x_n = x - x_n + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x}) dB_r - dV_s^{t,x},$$

that

$$E \left\{ \sup_{t \leq s \leq t_n} |X_s^{t,x} - x_n|^2 \right\} \leq C \left\{ |x - x_n|^2 + (1 + |x|^2) |t_n - t| + E \downarrow V^{t,x} \downarrow_{[t,t_n]}^2 \right\}. \quad (66)$$

Consequently, from (65) and (66), we have

$$E \sup_{t_n \leq s \leq T} |Y_s^{t_n, x_n} - Y_s^{t,x}|^2 \leq C \left\{ |x - x_n|^2 + (1 + |x|^2) |t_n - t| + E \downarrow V^{t,x} \downarrow_{[t,t_n]}^2 \right\}.$$

From (53), we see that $V^{t,x}$ is a process with continuous paths. But for any continuous bounded variation function g , we have $\downarrow g \downarrow_{[0,t]}$ is continuous in t . Thus, $\downarrow V^{t,x} \downarrow_{[t,t_n]} \rightarrow 0$, as $n \rightarrow \infty$, \mathbb{P} -a.s., and Dominated Convergence Theorem (recall (54), i.e., $\downarrow V^{t,x} \downarrow_{[t,T]} \in L^2(\Omega)$), yields that $E \downarrow V^{t,x} \downarrow_{[t,t_n]}^2 \rightarrow 0$, as $n \rightarrow \infty$. Thus, $E \sup_{t_n \leq s \leq T} |Y_s^{t_n, x_n} - Y_s^{t,x}|^2 \rightarrow 0$, as $n \rightarrow \infty$ and thanks to the L^2 -continuity of $Y^{t,x}$ we have $E |Y_{t_n}^{t,x} - Y_t^{t,x}|^2 \rightarrow 0$ as $n \rightarrow \infty$. Consequently,

$$\begin{aligned} |u(t_n, x_n) - u(t, x)|^2 &= E |Y_{t_n}^{t_n, x_n} - Y_t^{t,x}|^2 \leq 2E \sup_{t_n \leq s \leq T} |Y_s^{t_n, x_n} - Y_s^{t,x}|^2 + 2E |Y_{t_n}^{t,x} - Y_t^{t,x}|^2 \\ &\rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Step 2: Let us now show that u is left continuous w.r.t. t and continuous w.r.t. x . For this we consider $(t_n, x_n) \rightarrow (t, x)$ as $n \rightarrow \infty$, for $t_n \leq t$. We extend $X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, U_s^{t,x}$ to $s \in [t_n, T]$ by choosing $X_s^{t,x} = x, Y_s^{t,x} = Y_t^{t,x}, Z_s^{t,x} = 0, dV_s^{t,x} = x^* ds, U_s^{t,x} = u^*, s \in [t_n, t]$, for arbitrarily chosen $x^* \in \partial\psi(x)$ and $u^* \in \partial\varphi(Y_t^{t,x})$. Then we know that

$$\langle z - X_r^{t,x}, x^* \rangle + \psi(X_r^{t,x}) \leq \psi(z), \text{ for all } z \in \mathbb{R}^n, t_n \leq r \leq t, \text{ a.s.}$$

which yields that

$$\int_a^b \langle z - X_r^{t,x}, dV_r^{t,x} \rangle + \int_a^b \psi(X_r^{t,x}) dr \leq (b - a) \psi(z), \text{ for all } z \in \mathbb{R}^n, t_n \leq a \leq b \leq t, \text{ a.s.}$$

Moreover, it holds that $\int_a^b \langle z - X_r^{t,x}, dV_r^{t,x} \rangle + \int_a^b \psi(X_r^{t,x}) dr \leq (b - a) \psi(z)$, for all $z \in \mathbb{R}^n$, $t_n \leq a \leq b \leq T$, a.s. and $(Y_r^{t,x}, U_r^{t,x}) \in \partial\varphi$, $d\mathbb{P} \otimes dt$ a.e. on $\Omega \times [t_n, T]$.

Using the above extension, we have

$$\begin{cases} X_s^{t,x} + V_s^{t,x} = x + \int_{t_n}^s \tilde{b}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) ds + \int_{t_n}^s \tilde{\sigma}(r, X_r^{t,x}, Y_r^{t,x}) dB_r, \\ Y_s^{t,x} + \int_s^T U_r^{t,x} dr = g(X_T^{t,x}) + \int_s^T \tilde{f}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dB_r, \end{cases} s \in [t_n, T], \text{ a.s.}$$

where we define $\tilde{b}(r, x, y, z) = x^* 1_{[t_n, t]}(r) + b(r, x, y, z) 1_{[t, T]}(r)$, $\tilde{\sigma}(r, x, y) = \sigma(r, x, y) 1_{[t, T]}(r)$ and $\tilde{f}(r, x, y, z) = u^* 1_{[t_n, t]}(r) + f(r, x, y, z) 1_{[t, T]}(r)$.

From Proposition 14, there exists a constant C which does not depend on (t_n, x_n) such that

$$E \left\{ \sup_{t_n \leq s \leq T} |X_s^{t,x} - X_s^{t_n, x_n}|^2 + \sup_{t_n \leq s \leq T} |Y_s^{t,x} - Y_s^{t_n, x_n}|^2 + \int_{t_n}^T |Z_s^{t,x} - Z_s^{t_n, x_n}|^2 ds \right\} \leq C \Delta_3, \quad (67)$$

where

$$\begin{aligned} \Delta_3 &= e^{-\lambda t_n} |x - x_n|^2 + E \int_{t_n}^T |\tilde{b} - b|^2(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds \\ &\quad + E \int_{t_n}^T |\tilde{f} - f|^2(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds + E \int_{t_n}^T |\tilde{\sigma} - \sigma|^2(s, X_s^{t,x}, Y_s^{t,x}) ds \\ &= e^{-\lambda t_n} |x - x_n|^2 + E \int_{t_n}^t |x^* - b(s, X_t^{t,x}, Y_t^{t,x}, 0)|^2 ds \\ &\quad + E \int_{t_n}^t |u^* - f(s, X_t^{t,x}, Y_t^{t,x}, 0)|^2 ds + E \int_{t_n}^t |\sigma(s, X_t^{t,x}, Y_t^{t,x})|^2 ds. \end{aligned} \quad (68)$$

From Remark 15 we also have

$$\begin{aligned} &E \sup_{t \leq r \leq T} |X_r^{t,x}|^2 + E \sup_{t \leq r \leq T} |Y_r^{t,x}|^2 + E \int_t^T |Z_r^{t,x}|^2 dr \\ &\leq CE \left\{ |x|^2 + |g^0|^2 + \int_t^T |b^0(r)|^2 dr + \int_t^T |f^0(r)|^2 dr + \int_t^T |\sigma^0(r)|^2 dr \right\}. \end{aligned} \quad (69)$$

Hence, by combining (67)-(69) and considering that x^*, u^* only depends on t and x , we obtain that $E \sup_{t_n \leq s \leq T} |Y_s^{t,x} - Y_s^{t_n, x_n}|^2 \rightarrow 0$, as $n \rightarrow \infty$. Consequently,

$$|u(t_n, x_n) - u(t, x)|^2 = E |Y_{t_n}^{t_n, x_n} - Y_t^{t,x}|^2 \leq E \sup_{t_n \leq s \leq T} |Y_s^{t_n, x_n} - Y_s^{t,x}|^2 \rightarrow 0, \text{ as } n \rightarrow \infty. \quad \blacksquare$$

Remark 25 From (65), setting $t_n = t$, we know that $u(t, x)$ is Lipschitz continuous w.r.t. x on $[0, T] \times \text{Dom}\psi$.

Theorem 26 Suppose (A1) holds and $\text{Dom}\psi$ is locally compact, then the function $u(t, x) = Y_t^{t,x}$, $(t, x) \in [0, T] \times \text{Dom}\psi$ is a viscosity solution of PVI (1).

As a consequence of Theorem 4 and Theorem 26, we have

Theorem 27 We assume that $\text{Dom}\psi$ is locally compact, $\sigma(t, x, y)$ does not depend on y and f is Lipschitz continuous w.r.t. y . Then under the assumptions of (A1), PVI (1) has a unique viscosity solution in the class of functions which are Lipschitz continuous in x uniformly w.r.t. t and continuous in t .

Proof of Theorem 26. We use the following penalized equation to approximate (59):

$$\begin{cases} X_s^\varepsilon + \int_t^s \nabla \psi_\varepsilon(X_r^\varepsilon) dr = x + \int_t^s b(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) dr + \int_t^s \sigma(r, X_r^\varepsilon, Y_r^\varepsilon) dB_r, \\ Y_s^\varepsilon + \int_s^T \nabla \varphi_\varepsilon(Y_r^\varepsilon) dr = g(X_T^\varepsilon) + \int_s^T f(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) dr - \int_s^T Z_r^\varepsilon dB_r, \end{cases} \quad (70)$$

$s \in [t, T]$. From Theorem 30, we know that for any $x \in \mathbb{R}^n$, the above FBSDE has a unique solution $\{(X_s^{\varepsilon;t,x}, Y_s^{\varepsilon;t,x}, Z_s^{\varepsilon;t,x}), s \in [t, T]\}$. Putting

$$u^\varepsilon(t, x) := Y_t^{\varepsilon;t,x}, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^n,$$

We can argue similarly to Proposition 24 in order to show that u^ε is continuous on $[0, T] \times \mathbb{R}^n$. Moreover, with the help of Theorem 5.1 [21] we can prove that $u^\varepsilon(t, x)$ is the viscosity solution of the following backward quasilinear second-order parabolic PDE:

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial s}(s, x) + (\mathcal{L}u^\varepsilon)(s, x, u^\varepsilon(s, x), (\nabla u^\varepsilon(s, x))^* \sigma(s, x, u^\varepsilon(s, x))) \\ \quad + f(s, x, u^\varepsilon(s, x), (\nabla u^\varepsilon(s, x))^* \sigma(s, x, u^\varepsilon(s, x))) = \nabla \varphi_\varepsilon(u^\varepsilon(s, x)) + \langle \nabla \psi_\varepsilon(x), \nabla u^\varepsilon(s, x) \rangle, \\ \quad (s, x) \in [0, T] \times \mathbb{R}^n, \\ u^\varepsilon(T, x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \quad (71)$$

Similarly to the proof of Proposition 22 (ii), taking $\varepsilon_2 \rightarrow 0$, we can prove that

$$\begin{aligned} |u^\varepsilon(t, x) - u(t, x)|^2 &\leq E \sup_{t \leq s \leq T} |Y_s^{\varepsilon;t,x} - Y_s^{t,x}|^2 \\ &\leq C(1 + |x|^{3+\rho_0}) [1 + |x|^{3+\rho_0} + \psi^{1+2\rho}(x)] \varepsilon^{\frac{\rho}{2+4\rho}}, \text{ for all } (t, x) \in [0, T] \times \text{Dom}\psi. \end{aligned} \quad (72)$$

Here C is a constant which does not depend on (t, x) and ε .

Now we will show that u is a subsolution of PVI (1). From Lemma 6.1 [7] we know that for any point $(p, q, X) \in \mathcal{P}^{2,+}u(t, x)$, there exist sequences:

$$0 < \varepsilon_n \rightarrow 0, \quad (t_n, x_n) \rightarrow (t, x), \quad (p_n, q_n, X_n) \in \mathcal{P}^{2,+}u^{\varepsilon_n}(t_n, x_n),$$

such that

$$(p_n, q_n, X_n) \rightarrow (p, q, X).$$

For any n , using that $(p_n, q_n, X_n) \in \mathcal{P}^{2,+}u^{\varepsilon_n}(t_n, x_n)$, we obtain

$$\begin{aligned} -p_n - \frac{1}{2} \text{Tr}(\sigma \sigma^*(t_n, x_n, u^{\varepsilon_n}(t_n, x_n)) X_n) - \langle b(t_n, x_n, u^{\varepsilon_n}(t_n, x_n), q_n^* \sigma(t_n, x_n, u^{\varepsilon_n}(t_n, x_n))), q_n \rangle \\ - f(t_n, x_n, u^{\varepsilon_n}(t_n, x_n), q_n^* \sigma(t_n, x_n, u^{\varepsilon_n}(t_n, x_n))) \leq -\nabla \varphi_{\varepsilon_n}(u^{\varepsilon_n}(t_n, x_n)) - \langle \nabla \psi_{\varepsilon_n}(x_n), q_n \rangle. \end{aligned} \quad (73)$$

We can assume that $u(t, x) > \inf(\text{Dom}\varphi)$. Indeed, if we have $u(t, x) = \inf(\text{Dom}\varphi)$, then $\varphi'_-(u(t, x)) = -\infty$, and inequality (2) would hold obviously. Let $y \in \text{Dom}\varphi, y < u(t, x)$. Since u^ε converges to u uniformly on compacts (see (72)), there exists $n_0 \in \mathbb{N}$ s.t. $y < u^{\varepsilon_n}(t_n, x_n)$, $n > n_0$. Multiplying $u^{\varepsilon_n}(t_n, x_n) - y$ with both sides of (73) and using

$$\varphi(J_{\varepsilon, \varphi} x) \leq \varphi_\varepsilon(x) \leq \varphi_\varepsilon(z) + (x - z, \nabla \varphi_\varepsilon(x)) \leq \varphi(z) + (x - z, \nabla \varphi_\varepsilon(x)), \quad z \in \mathbb{R},$$

we obtain

$$\begin{aligned}
& \left[-p_n - \frac{1}{2}Tr(\sigma\sigma^*(t_n, x_n, u^{\varepsilon_n}(t_n, x_n))X_n) - \langle b(t_n, x_n, u^{\varepsilon_n}(t_n, x_n), q_n^*\sigma(t_n, x_n, u^{\varepsilon_n}(t_n, x_n))), q_n \rangle \right. \\
& \quad \left. - f(t_n, x_n, u^{\varepsilon_n}(t_n, x_n), q_n^*\sigma(t_n, x_n, u^{\varepsilon_n}(t_n, x_n))) \right] \left[u^{\varepsilon_n}(t_n, x_n) - y \right] \\
& \quad + \langle \nabla\psi_{\varepsilon_n}(x_n), q_n \rangle \left[u^{\varepsilon_n}(t_n, x_n) - y \right] + \varphi(J_{\varepsilon_n, \varphi}u^{\varepsilon_n}(t_n, x_n)) \leq \varphi(y).
\end{aligned} \tag{74}$$

Let us take now $\liminf_{n \rightarrow \infty}$ in the above inequality. Recalling (13-c), $\nabla\psi_{\varepsilon_n}(x_n) \in \partial\psi(J_{\varepsilon_n, \psi}(x_n))$, $J_{\varepsilon_n, \psi}(x_n) \rightarrow x$ and $J_{\varepsilon_n, \varphi}(u^{\varepsilon_n}(t_n, x_n)) \rightarrow u(t, x)$, the lower limit in (74) yields

$$\begin{aligned}
& \left[-p - \frac{1}{2}Tr(\sigma\sigma^*(t, x, u(t, x))X) - \langle b(t, x, u(t, x), q^*\sigma(t, x, u(t, x))), q \rangle \right. \\
& \quad \left. - f(t, x, u(t, x), q^*\sigma(t, x, u(t, x))) \right] \left[u(t, x) - y \right] \\
& \quad + \partial\psi_*(x, q) \left[u(t, x) - y \right] + \varphi(u(t, x)) \leq \varphi(y).
\end{aligned}$$

Then

$$\begin{aligned}
& -p - \frac{1}{2}Tr(\sigma\sigma^*(t, x, u(t, x))X) - \langle b(t, x, u(t, x), q^*\sigma(t, x, u(t, x))), q \rangle \\
& \quad - f(t, x, u(t, x), q^*\sigma(t, x, u(t, x))) \leq -\frac{\varphi(u(t, x)) - \varphi(y)}{u(t, x) - y} - \partial\psi_*(x, q), \text{ for all } y < u(t, x),
\end{aligned}$$

and taking the limit $y \rightarrow u(t, x)$ yields (2). Therefore u is a viscosity subsolution of PVI (1). Similarly, we prove that u is a viscosity supersolution of PVI (1). \blacksquare

6 Appendix

6.1 A priori estimates for penalized FBSDEs

In this subsection, following the ideas of [9], we give a priori estimates for the solution of the following penalized FBSDE:

$$\begin{cases} X_s^\varepsilon + \int_t^s \nabla\psi_\varepsilon(X_r^\varepsilon)dr = \xi + \int_t^s b(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon)dr + \int_t^s \sigma(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon)dB_r, \\ Y_s^\varepsilon + \int_s^T \nabla\varphi_\varepsilon(Y_r^\varepsilon)dr = g(X_T^\varepsilon) + \int_s^T f(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon)dr - \int_s^T Z_r^\varepsilon dB_r, \quad s \in [t, T], \end{cases} \tag{75}$$

where $t \in [0, T)$, $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$ and the coefficients are supposed to satisfy $(H_1) - (H_5)$.

Lemma 28 *Assume $(H_1) - (H_5)$ hold. Suppose that $(X^{\varepsilon, t, \xi_i}, Y^{\varepsilon, t, \xi_i}, Z^{\varepsilon, t, \xi_i})$ is the solution of FBSDE (75) with $\xi = \xi_i \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$ for both $i = 1, 2$. Let us put $\hat{\xi} = \xi_1 - \xi_2$, $\hat{X}^\varepsilon = X^{\varepsilon, t, \xi_1} - X^{\varepsilon, t, \xi_2}$, $\hat{Y}^\varepsilon = Y^{\varepsilon, t, \xi_1} - Y^{\varepsilon, t, \xi_2}$ and $\hat{Z}^\varepsilon = Z^{\varepsilon, t, \xi_1} - Z^{\varepsilon, t, \xi_2}$. Then, for arbitrary $\lambda \in \mathbb{R}$ and arbitrary positive constants C_1, C_2, C_3, C_4 , we have*

$$\begin{aligned}
e^{-\lambda T} E|\hat{X}_T^\varepsilon|^2 + \bar{\lambda}_1 \|\hat{X}^\varepsilon\|_{M_\lambda[t, T]}^2 & \leq e^{-\lambda t} E|\hat{\xi}|^2 + K(C_1 + K) \|\hat{Y}^\varepsilon\|_{M_\lambda[t, T]}^2 \\
& \quad + (KC_2 + k_1^2) \|\hat{Z}^\varepsilon\|_{M_\lambda[t, T]}^2.
\end{aligned} \tag{76}$$

Furthermore, if in addition $KC_4 = 1 - \alpha$ for some $0 < \alpha < 1$, then

$$\|\hat{Y}^\varepsilon\|_{M_\lambda[t,T]}^2 \leq B(\bar{\lambda}_2, T-t) \left[k_2^2 e^{-\lambda T} E|\hat{X}_T^\varepsilon|^2 + KC_3 \|\hat{X}^\varepsilon\|_{M_\lambda[t,T]}^2 \right], \quad (77)$$

$$\|\hat{Z}^\varepsilon\|_{M_\lambda[t,T]}^2 \leq \frac{A(\bar{\lambda}_2, T-t)}{\alpha} \left[k_2^2 e^{-\lambda T} E|\hat{X}_T^\varepsilon|^2 + KC_3 \|\hat{X}^\varepsilon\|_{M_\lambda[t,T]}^2 \right]. \quad (78)$$

Here $\bar{\lambda}_1 = \lambda - K(2 + C_1^{-1} + C_2^{-1}) - K^2$ and $\bar{\lambda}_2 = -\lambda - 2\gamma - K(C_3^{-1} + C_4^{-1})$. Recall that A and B are defined in (12) as $A(\lambda, s) = e^{-(\lambda \wedge 0)s}$ and $B(\lambda, s) = \int_0^s e^{-\lambda r} dr$.

Proof. We apply Itô's formula to $\left(e^{-\lambda r} e^{-\lambda'(s-r)} |\hat{X}_r^\varepsilon|^2 \right)_{r \in [t,s]}$ and $\left(e^{-\lambda r} e^{-\lambda'(r-s)} |\hat{Y}_r^\varepsilon|^2 \right)_{r \in [s,T]}$. Thanks to the convexity of ψ and φ , we have $\langle x_1 - x_2, \nabla \psi_\varepsilon(x_1) - \nabla \psi_\varepsilon(x_2) \rangle \geq 0$ and $\langle y_1 - y_2, \nabla \varphi_\varepsilon(y_1) - \nabla \varphi_\varepsilon(y_2) \rangle \geq 0$, which allow us to repeat the argument of Lemma 5.1 [9] to obtain our result. \blacksquare

We recall that we have introduced the notations $b^0(s) := b(\cdot, s, 0, 0, 0)$, $\sigma^0(s) := \sigma(\cdot, s, 0, 0, 0)$, $f^0(s) := f(\cdot, s, 0, 0, 0)$, $g^0 := g(\cdot, 0)$.

Proposition 29 *Let the assumptions (H_1) – (H_5) be satisfied. We also assume $(C1)$ and either $(C2)$ or $(C3)$ hold for some $\lambda, \alpha, C_1, C_2, C_3$, and $C_4 = \frac{1-\alpha}{K}$. Then for $t \in [0, T]$ and $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, there exists a constant C independent of ε and the initial data (t, ξ) , such that*

$$\|X^{\varepsilon, t, \xi}\|_{S_\lambda[t,T]}^2 + \|Y^{\varepsilon, t, \xi}\|_{S_\lambda[t,T]}^2 + \|Z^{\varepsilon, t, \xi}\|_{M_\lambda[t,T]}^2 \leq C\Gamma_1, \quad (79)$$

where

$$\Gamma_1 = e^{-\lambda t} E|\xi|^2 + \|b^0\|_{M_\lambda[t,T]}^2 + \|\sigma^0\|_{M_\lambda[t,T]}^2 + \|f^0\|_{M_\lambda[t,T]}^2 + E|g^0|^2.$$

Moreover, if among the compatibility conditions, only $(C1)$ holds, then there exists a constant $T_0 > 0$ small enough (depending only on K, γ, k_1, k_2) such that for all $t \in [0, T]$ with $T-t \leq T_0$, we have the above estimates.

Proof. Similarly to Lemma 5.2 [9], using $A(\lambda, T-t) \leq A(\lambda, T)$, $B(\lambda, T-t) \leq B(\lambda, T)$, $\langle x_1 - x_2, \nabla \psi_\varepsilon(x_1) - \nabla \psi_\varepsilon(x_2) \rangle \geq 0$ and $\langle y_1 - y_2, \nabla \varphi_\varepsilon(y_1) - \nabla \varphi_\varepsilon(y_2) \rangle \geq 0$, we get, for arbitrary $\lambda \in \mathbb{R}$, $\delta > 0$ and positive constants C_1, C_2, C_3, C_4

$$\begin{aligned} e^{-\lambda T} E|X_T^{\varepsilon, t, \xi}|^2 + \bar{\lambda}_1^\delta \|X^{\varepsilon, t, \xi}\|_{M_\lambda[t,T]}^2 &\leq K[C_1 + K(1 + \delta)] \|Y^{\varepsilon, t, \xi}\|_{M_\lambda[t,T]}^2 \\ &+ [KC_2 + k_1^2(1 + \delta)] \|\hat{Z}^{\varepsilon, t, \xi}\|_{M_\lambda[t,T]}^2 + e^{-\lambda t} E|\xi|^2 + \frac{1}{\delta} \|b^0\|_{M_\lambda[t,T]}^2 + (1 + \frac{1}{\delta}) \|\sigma^0\|_{M_\lambda[t,T]}^2. \end{aligned} \quad (80)$$

Furthermore, if in addition, $KC_4 = 1 - \alpha$, for some $0 < \alpha < 1$, then

$$\begin{aligned} \|Y^{\varepsilon, t, \xi}\|_{M_\lambda[t,T]}^2 &\leq B(\bar{\lambda}_2^\delta, T) \left\{ k_2^2(1 + \delta) e^{-\lambda T} E|X_T^{\varepsilon, t, \xi}|^2 + KC_3 \|X^{\varepsilon, t, \xi}\|_{M_\lambda[t,T]}^2 \right. \\ &\quad \left. + \frac{1}{\delta} \|f^0\|_{M_\lambda[t,T]}^2 + (1 + \frac{1}{\delta}) e^{-\lambda T} E|g^0|^2 \right\}, \end{aligned} \quad (81)$$

$$\begin{aligned} \|Z^{\varepsilon, t, \xi}\|_{M_\lambda[t,T]}^2 &\leq \frac{A(\bar{\lambda}_2^\delta, T)}{\alpha} \left\{ k_2^2(1 + \delta) e^{-\lambda T} E|X_T^{\varepsilon, t, \xi}|^2 + KC_3 \|X^{\varepsilon, t, \xi}\|_{M_\lambda[t,T]}^2 \right. \\ &\quad \left. + \frac{1}{\delta} \|f^0\|_{M_\lambda[t,T]}^2 + (1 + \frac{1}{\delta}) e^{-\lambda T} E|g^0|^2 \right\}, \end{aligned} \quad (82)$$

where $\bar{\lambda}_1^\delta = \bar{\lambda}_1 - (1 + K^2)\delta$ and $\bar{\lambda}_2^\delta = \bar{\lambda}_2 - \delta$. Now we define

$$\mu^\delta(\alpha, T) := K[C_1 + K(1 + \delta)]B(\bar{\lambda}_2^\delta, T) + \frac{A(\bar{\lambda}_2^\delta, T)}{\alpha}[KC_2 + k_1^2(1 + \delta)].$$

Then plugging (81) and (82) into (80) yields

$$\begin{aligned} & (1 - \mu^\delta(\alpha, T)k_2^2(1 + \delta))e^{-\lambda T}E|X_T^{\varepsilon, t, \xi}|^2 + (\bar{\lambda}_1^\delta - \mu^\delta(\alpha, T)KC_3)\|X^{\varepsilon, t, \xi}\|_{M_\lambda[t, T]}^2 \\ & \leq e^{-\lambda t}E|\xi|^2 + \frac{1}{\delta}\|b^0\|_{M_\lambda[t, T]}^2 + (1 + \frac{1}{\delta})\|\sigma^0\|_{M_\lambda[t, T]}^2 \\ & \quad + \mu^\delta(\alpha, T)\left\{\frac{1}{\delta}\|f^0\|_{M_\lambda[t, T]}^2 + (1 + \frac{1}{\delta})e^{-\lambda T}E|g^0|^2\right\}. \end{aligned} \quad (83)$$

Observe that $\bar{\lambda}_1^\delta \rightarrow \bar{\lambda}_1$ and $\mu^\delta(\alpha, T) \rightarrow \mu(\alpha, T)$ as $\delta \rightarrow 0$ (Recall the definition of $\mu(\alpha, T)$ in (12)). On the other hand, due to the assumption (C2), there exists $\alpha \in (0, 1)$ s.t. $\mu(\alpha, T)KC_3 < \bar{\lambda}_1$. Consequently, we can choose a sufficiently small $\delta > 0$ which is independent of ε and t , such that $\bar{\lambda}_1^\delta - \mu^\delta(\alpha, T)KC_3 > 0$. Then we obtain for our fixed $\delta > 0$ from (83),

$$\|X^{\varepsilon, t, \xi}\|_{M_\lambda[t, T]}^2 \leq C\Gamma_1,$$

where

$$C = \frac{(1 + \frac{1}{\delta})(1 + \mu^\delta(\alpha, T))}{\bar{\lambda}_1^\delta - \mu^\delta(\alpha, T)KC_3},$$

and

$$\Gamma_1 = e^{-\lambda t}E|\xi|^2 + \|b^0\|_{M_\lambda[t, T]}^2 + \|\sigma^0\|_{M_\lambda[t, T]}^2 + \|f^0\|_{M_\lambda[t, T]}^2 + E|g^0|^2.$$

Similarly, under the assumptions (C1) and (C3), we know that there exists $\alpha \in (k_1^2 k_2^2, 1)$, such that $\mu(\alpha, T)k_2^2 < 1$ and $\bar{\lambda}_1 \geq \frac{KC_3}{k_2^2}$. Hence, we can choose $\delta > 0$ such that $1 - \mu^\delta(\alpha, T)k_2^2(1 + \delta) > 0$ and $\bar{\lambda}_1^\delta - \mu^\delta(\alpha, T)KC_3 \geq \bar{\lambda}_1^\delta - \bar{\lambda}_1\mu^\delta(\alpha, T)k_2^2 > 0$. Then from (83) it follows

$$e^{-\lambda T}E|X_T^{\varepsilon, t, \xi}| + \|X^{\varepsilon, t, \xi}\|_{M_\lambda[t, T]}^2 \leq C\Gamma_1,$$

where

$$C = \max \left\{ \frac{(1 + \frac{1}{\delta})(1 + \mu^\delta(\alpha, T))}{1 - \mu^\delta(\alpha, T)k_2^2(1 + \delta)}, \frac{(1 + \frac{1}{\delta})(1 + \mu^\delta(\alpha, T))}{\bar{\lambda}_1^\delta - \mu^\delta(\alpha, T)KC_3} \right\}.$$

Using (81) and (82), the above estimates for $X^{\varepsilon, t, \xi}$ allows to deduce that

$$\|X^{\varepsilon, t, \xi}\|_{M_\lambda[t, T]}^2 + \|Y^{\varepsilon, t, \xi}\|_{M_\lambda[t, T]}^2 + \|Z^{\varepsilon, t, \xi}\|_{M_\lambda[t, T]}^2 \leq C\Gamma_1.$$

Moreover, applying Itô's formula to $|X_r^{\varepsilon, t, \xi}|^2$ and $|Y_r^{\varepsilon, t, \xi}|^2$, and using (13-d)), the BDG and the Hölder inequality, we obtain $\|X^{\varepsilon, t, \xi}\|_{S_\lambda[t, T]}^2 + \|Y^{\varepsilon, t, \xi}\|_{S_\lambda[t, T]}^2 \leq C\Gamma_1$.

Finally, if only (C1) holds, similar to the argument of [9], if $k_2 = 0$, for fixed C_1, C_2, C_3 and $\alpha \in (0, 1)$, we have $\mu(\alpha, 0)KC_3 = \frac{(KC_2 + k_1^2)KC_3}{\alpha} \leq \bar{\lambda}_1$, for λ big enough. Then by the continuity of $\mu(\alpha, \cdot)$, we can find a $T_0 > 0$ small enough and only depending on K, γ, k_1, k_2 such that $\mu(\alpha, T - t)KC_3 \leq \bar{\lambda}_1$ for $T - t \leq T_0$. If $k_2 > 0$, we choose $\alpha \in (k_1^2 k_2^2, 1)$, $C_2 = \frac{\alpha - k_1^2 k_2^2}{4Kk_2^2}$ and $C_4 = \frac{1 - \alpha}{K}$. Then we have $\mu(\alpha, 0) = \frac{KC_2 + k_1^2}{\alpha} < \frac{1}{k_2^2}$ and $\bar{\lambda}_1 \geq \frac{KC_3}{k_2^2}$, for λ big enough. Consequently, there exists a $T_0 > 0$ small enough (only depending on K, γ, k_1, k_2) such that

$\mu(\alpha, T-t) < \frac{1}{k_2^2}$ and $\bar{\lambda}_1 \geq \frac{KC_3}{k_2^2}$, for $T-t \leq T_0$. Reproducing the above argument, we obtain the same estimates when (C1) holds and $T-t \leq T_0$, for some T_0 small enough. ■

Now we give the solvability result for FBSDE (75).

Theorem 30 *Let the assumptions (H_1) – (H_5) be satisfied. We also assume (C1) and either (C2) or (C3) hold for some $\lambda, \alpha, C_1, C_2, C_3$ and $C_4 = \frac{1-\alpha}{K}$. Then the penalized FBSDE (75) has a unique adapted solution $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon) \in S_n^2 \times S_m^2 \times M_{m \times d}^2$. Moreover, if among the compatibility conditions, only (C1) holds, then there exists a constant $T_0 > 0$ small enough (depending on K, γ, k_1, k_2) such that for all $t \in [0, T]$ with $T-t \leq T_0$, the penalized FBSDE (75) has a unique adapted solution on $[t, T]$.*

Proof. We introduce a mapping $\Lambda : H \rightarrow H$ (Recalling the definition of $H = H[0, T]$) such that $\bar{X}_s^\varepsilon := \Lambda(X^\varepsilon)_s$ ($X^\varepsilon \in H$) is the unique solution of the SDE:

$$\bar{X}_s^\varepsilon + \int_t^s \nabla \psi_\varepsilon(\bar{X}_r^\varepsilon) dr = \xi + \int_t^s b(r, \bar{X}_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) dr + \int_t^s \sigma(r, \bar{X}_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) dB_r,$$

where $(Y^\varepsilon, Z^\varepsilon)$ is the unique solution of the following BSDE

$$Y_s^\varepsilon + \int_s^T \nabla \varphi_\varepsilon(Y_r^\varepsilon) dr = g(X_T^\varepsilon) + \int_s^T f(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) dr - \int_s^T Z_r^\varepsilon dB_r.$$

Given arbitrary $X^\varepsilon, \tilde{X}^\varepsilon \in H$, we put $\Delta X^\varepsilon = X^\varepsilon - \tilde{X}^\varepsilon$ and $\Delta \bar{X}^\varepsilon = \Lambda(X^\varepsilon) - \Lambda(\tilde{X}^\varepsilon)$. The estimates (76)–(78) yield

$$\begin{aligned} & e^{-\lambda T} E|\Delta \bar{X}_T^\varepsilon|^2 + \bar{\lambda}_1 \|\Delta \bar{X}^\varepsilon\|_{M_\lambda[t, T]}^2 \\ & \leq \mu(\alpha, T-t) \left\{ k_2^2 e^{-\lambda T} E|\Delta X_T^\varepsilon|^2 + KC_3 \|\Delta X^\varepsilon\|_{M_\lambda[t, T]}^2 \right\}. \end{aligned} \quad (84)$$

With a similar discussion as in Theorem 3.1 [9] we can prove Λ is a contracting mapping on $(M_n^2[t, T], \|\cdot\|_{M_\lambda[t, T]})$, if (C1) and (C2) hold, and Λ is a contracting mapping on $(\bar{H}[t, T], \|\cdot\|_{\lambda, \beta, [t, T]})$, if (C1) and (C3) hold. Moreover, if only (C1) holds and $k_2 = 0$ (resp. $k_2 > 0$), Λ is a contracting mapping on $(M_n^2[t, T], \|\cdot\|_{M_\lambda[t, T]})$ (resp. $(\bar{H}[t, T], \|\cdot\|_{\lambda, \beta, [t, T]})$), for all $t \in [0, T]$ with $T-t \leq T_0$, $T_0 > 0$ small enough. ■

Similar to the proof Theorem A.5 [10], using $\langle x_1 - x_2, \nabla \psi_\varepsilon(x_1) - \nabla \psi_\varepsilon(x_2) \rangle \geq 0$ and $\langle y_1 - y_2, \nabla \varphi_\varepsilon(y_1) - \nabla \varphi_\varepsilon(y_2) \rangle \geq 0$, we have the following proposition:

Proposition 31 *Suppose $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ (resp. $(\tilde{X}^\varepsilon, \tilde{Y}^\varepsilon, \tilde{Z}^\varepsilon)$) is a solution of FBSDE (75) with parameters (ξ, b, σ, f, g) (resp. $(\tilde{\xi}, \tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g})$) which satisfy (H_1) – (H_5) and $(H'_2), (H'_3)$ and $\xi, \tilde{\xi} \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$. Let $1 \leq p \leq \frac{3+\rho_0}{2}$, such that*

$$E \left\{ \sup_{t \leq s \leq T} |X_s^\varepsilon|^{2p} + \sup_{t \leq s \leq T} |Y_s^\varepsilon|^{2p} + \left(\int_t^T |Z_s^\varepsilon|^2 ds \right)^p \right\} < \infty,$$

and

$$E \left\{ \sup_{t \leq s \leq T} |\tilde{X}_s^\varepsilon|^{2p} + \sup_{t \leq s \leq T} |\tilde{Y}_s^\varepsilon|^{2p} + \left(\int_t^T |\tilde{Z}_s^\varepsilon|^2 ds \right)^p \right\} < \infty.$$

Then there exist constants $\delta_{K,k_2,\gamma,p} > 0$ and $C_{K,k_2,\gamma,p}$ depending only on K, k_2, γ, p , such that, for all $t \in [0, T]$ with $T - t \leq \delta_{K,k_2,\gamma,p}$,

$$\begin{aligned} & E \left\{ \sup_{t \leq s \leq T} |\tilde{X}_s^\varepsilon - X_s^\varepsilon|^{2p} + \sup_{t \leq s \leq T} |\tilde{Y}_s^\varepsilon - Y_s^\varepsilon|^{2p} + \left(\int_t^T |\tilde{Z}_s^\varepsilon - Z_s^\varepsilon|^2 ds \right)^p \right\} \\ & \leq C_{K,k_2,\gamma,p} E \left\{ |\tilde{\xi} - \xi|^{2p} + |\tilde{g} - g|^{2p}(X_T^\varepsilon) + \left(\int_t^T |\tilde{\sigma} - \sigma|^2(s, X_s^\varepsilon, Y_s^\varepsilon) ds \right)^p \right. \\ & \quad \left. + \left(\int_t^T (|\tilde{b} - b| + |\tilde{f} - f|)(s, X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon) ds \right)^{2p} \right\}. \end{aligned} \quad (85)$$

6.2 Auxiliary Propositions

In this section, for the reader's convenience, we give two auxiliary propositions (see [20]).

Proposition 32 (Remark 2.34 [20]) *Let X_t be an n -dimensional Itô process given by*

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s, \quad t \in [0, T],$$

where b, σ satisfies $E \left(\int_0^T |b_s| ds + \int_0^T |\sigma_s|^2 ds \right) < \infty$. Assume $h \in C^1(\mathbb{R}^n; \mathbb{R})$ and that there exists a constant M such that

$$|\nabla_x h(x + y) - \nabla_x h(x)| \leq M|y|, \quad \text{for all } x, y \in \mathbb{R}^n. \quad (86)$$

Then, for all $0 \leq s \leq t$, $\mathbb{P} - a.s.$

$$h(X_t) \leq h(X_s) + \int_s^t \langle \nabla_x h(X_r), dX_r \rangle + \frac{1}{2} \int_s^t M |\sigma_r|^2 dr.$$

Remark 33 *Similarly, for $r > 0$ and $h \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ satisfying (86), we can prove that*

$$\begin{aligned} h^{1+2r}(X_t) & \leq h^{1+2r}(X_s) + (1+2r) \int_s^t h^{2r}(X_r) \langle \nabla_x h(X_r), dX_r \rangle \\ & \quad + (1+2r)r \int_s^t h^{2r-1}(X_r) |\nabla_x h(X_r)|^2 |\sigma_r|^2 dr + \frac{1+2r}{2} \int_s^t h^{2r}(X_r) M |\sigma_r|^2 dr, \quad \text{for all } 0 \leq s \leq t. \end{aligned} \quad (87)$$

Proposition 34 (Proposition 1.25, [20]) *Let (X, K) , (X^n, K^n) , $n \geq 1$, be a sequence of couples of $C([0, T]; \mathbb{R}^d)$ -valued random variables such that*

- (i) *There is $p > 0$, with $\sup_{n \geq 1} E \uparrow K^n \downarrow_{[0,T]}^p < \infty$,*
- (ii) *$\sup_{0 \leq t \leq T} |X_t^n - X_t| + \sup_{0 \leq t \leq T} |K_t^n - K_t| \xrightarrow{\mathbb{P}} 0$, as $n \rightarrow \infty$.*

Then, for all $0 \leq s \leq t \leq T$,

$$\int_s^t \langle X_r^n, dK_r^n \rangle \xrightarrow{\mathbb{P}} \int_s^t \langle X_r, dK_r \rangle, \text{ as } n \rightarrow \infty,$$

and, moreover,

$$E \uparrow K \uparrow_{[0,T]}^p \leq \liminf_{n \rightarrow \infty} E \uparrow K^n \uparrow_{[0,T]}^p .$$

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